Department of Education, Ontario

Annual Examinations, 1943

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. If n is a positive integer, prove that

$$(1-x)^{3n} + 3nx(1-x)^{3n-2} + \frac{3n(3n-3)}{1\cdot 2}x^2(1-x)^{3n-4} + \cdots = (1-x^3)^n \ .$$

2. Solve the system:

$$31x^2y^2 - 7y^4 - 112xy + 64 = 0$$

$$x^2 - 7xy + 4y^2 + 8 = 0 \ .$$

3. If the quadratic function $3x^2 + 2pxy + 2y^2 + 2ax - 4y + 1$ can be resolved into factors linear in x and y, prove that p must be a root of the equation $p^2 + 4ap + 2a^2 + 6 = 0$.

4. Sum to n terms the series whose nth term is

$$\frac{n^4 + 2n^3 + n^2 - 1}{n^2 + n} \ .$$

5. If C and D are two points on one branch of a hyperbola whose foci are A and B, prove that, in general, A and B are on one branch of a hyperbola whose foci are C and D.

   Examine the case where C and D are on opposite branches of the hyperbola.

6. Starting at the vertex A of the parabola $y^2 = 4px$, whose axis points east from A, successive chords AB, BC, CD, *etc.*, are drawn pointing laternately north-east (AB,CD,$\cdots$) and south-east ($BC, \cdots$). Find the co-ordinates of the end of the nth chord.

7. Show that the locus of intersections of tangents to an ellipse $ax^2 + by^2 = 1$ at the ends of perpendicular diameters is the ellipse $a^2x^2 + b^2y^2 = a + b$.

8. On a fixed line are three fixed points AQ, B, and C. A line AQR turns about A, while Q and R remain each at a fixed distance from A. Find the locus of the intersection of BQ and CR.

9. From a point c feet above the surface of a lake the angle of elevation of a cloud is α, and the angle of depression of its reflection in the lake is β. Show that the height of the cloud above the lake is $c\sin(\beta+\alpha)\cos(\beta-\alpha)$ feet.

10. If O_1, O_2, O_3 are the centres of the three escribed circles of a triangle ABC, prove that the area of the triangle $O_1O_2O_3$ is

$$\Delta\left(1+\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\right)$$

where Δ is the area of triangle ABC.

11. A triangle ABC is such that $3AB = 2AC$. Also a point D on BC is such that $BD = 2DC$ and $AD = BC$. Show that

$$\tan\frac{\angle ADB}{2} = \sqrt{\frac{5}{19}}\ .$$

12. If A, B, C are the angles of a triangle, prove that

$$1 < \cos A + \cos B + \cos C \leq \frac{3}{2}\ .$$