Department of Education, Ontario

Annual Examinations, 1958

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. A semicircle is described on the line AB as diameter. A rectangle with base on AQB is inscribed in the semicircle. If AB has length 10 inches, and x denotes the base of the rectangle, find x so that the area of the rectangle is a maximum. What is this maximum?

2. A uniform cylindrical tub can be filled with water from the cold water tap in 14 minutes and drained completely through a hole in the bottom in 21 minutes. When both hot and cold water taps are turned on and the plug is removed from the hole, the tub can be filled in 12.6 minutes. Find the time required to fill the tub from the hot water tap alone. (Assume constant flow through each opening.)

3. A sequence of numbers is defined by $x_1 = 2$, $x_{n+1} = 1 + \sqrt{5 + x_n}$ for $n \geq 1$. Prove

   (a) $x_n < x_{n+1}$ for all n;

   (b) $x_n < 4$ for all n;

   (c) as n becomes very large. x_n aproaches a limit α;

   (d) $\alpha = 4$.

4. Prove that, if a and b are integers, then the rational fraction a/b is either a terminating or a repeating decimal.

5. A line which passes through $(2, 2)$ and a line which passes through $(-2, -2)$ have $x-$intercepts which always differ by 4. Find the locus of the point of intersection of the two lines.

6. (a) Prove that the line with equation $y = mx + \sqrt{a^2m^2 - b^2}$ is tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ for all values of m.

(b) Find the locus of all points from which tangents to the hyperbola in (a) are perpendicular.

(c) Under what circumstances is there (i) only one such point, (ii) none?

7. If an ellipse and a hyperbola have the same foci, show that they intersect at right angles.

8. A straight line inclined at an angle of θ to the axis of x touches both the curves $y^2 = 4ax$ and $x^2 + y^2 = r^2$. Find the value of θ.

9. Solve the following system:

$$\cos x + \cos y = \cos z \ ,$$

$$\cos 2x + \cos 2y = \cos 2z \ ,$$

$$\cos 3x + \cos 3y = \cos 3z \ .$$

10. (a) Prove that

$$\cot^{-1}(2n^2) = \cot^{-1}(2n-1) - \cot^{-1}(2n+1) \ .$$

(b) Considering only the principal values of the terms of the series

$$\cot^{-1}(2 \cdot 1^2) + \cot^{-1}(2 \cdot 2^2) + \cot^{-1}(2 \cdot 3^2) + \cdots \ ,$$

find the sum to n terms and the sum to infinity.

11. (a) Prove that $\cos A \cos B \leq \cos^2(\frac{A+B}{2})$.

(b) For any triangle ABC, prove that

$$\cos A \cos B \cos C \leq \frac{1}{8} \ .$$

12. A uniform bar AB has one end A in contact with a rough horizontal floor and is kept in equilibrium at an angle θ to the floor by its own weight and a force applied at B perpendicular to AB. Prove that the coefficient of friction between the bar and the floor is given by

$$\mu = \frac{1}{2\tan\theta + \cot\theta} \ .$$