PUTNAM PROBLEMS

REAL NUMBERS

2009-A-4. Let S be a set of rational numbers such that
(a) $0 \in S$;
(b) If $x \in S$, then $x + 1 \in S$ and $x - 1 \in S$;
(c) If $x \in S$ and $x \notin (0, 1)$, then $1/(x(x - 1)) \in S$.

Must S contain all rational numbers?

2009-B-2. A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

2008-B-1. What is the maximum number of rational points that can be on a circle in $\mathbf{R}^2$ whose centre is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

2006-B-2. Prove that, for every set $X = \{x_1, x_2, \cdots, x_n\}$ of real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1} \; .$$

2006-B-6. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n \geq 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k} \; .$$

2003-B-6. Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geq \int_0^1 |f(x)| dx \; .$$

2002-B-3. Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne} \; .$$

2000-A-1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, x_2, \cdots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

1998-B-5. Let N be a positive integer with 1998 decimal digits, all of them 1; that is, $N = 1111 \cdots 11$ (1998 digits). Find the thousandth digits after the decimal point of $\sqrt{N}$.

1997-B-1. Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$S_n = \sum_{m=1}^{6n-1} \min \left(\left\{ \frac{m}{6n} \right\}, \left\{ \frac{m}{3n} \right\} \right) \; .$$

(Here, min (a, b) denotes the minimum of a and b.)

1995-A-1. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (notnecessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

1995-B-6. For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \cdots\}.$$

Prove that $\{1, 2, 3, \cdots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$. [As usual $\lfloor x \rfloor$ is the greatest integer $\leq x$.]

1994-A-5. Let $(r_n)_{n\geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{r_{1994}} \ , \qquad \text{with} \ \ i_1 < i_2 < \cdots < i_{1994} \ \ .$$

Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

1990-A-2. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n, m = 0, 1, 2, \cdots)$? Justify your answer.

1990-A-4. Consider a paper punch that can be centered at any point of the plane, and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?