Quadratic inverse example

Let $f(x) = x^2 + 3x + 2 = (x - \frac{3}{2})^2 + \frac{1}{4}$. The equation $x^2 + 3x + 2 = y$ has the solution $x = \frac{1}{2}(-3 \pm \sqrt{1+4y})$ so the (composition) inverse function $g(x) = f^{-1}(x)$ is apparently given by

$$g(x) = \frac{-3 \pm \sqrt{1+4x}}{2}.$$

Let us check this and see what happens.

$$f(g(x)) = (g(x))^2 + 3g(x) + 2 = \left[\frac{5 \mp 3\sqrt{1+4x}}{2}\right] + x + \left[\frac{-9 \pm 3\sqrt{1+4x}}{2}\right] + 2 = x.$$

So far, so good. Now compose the two functions in the opposite order:

$$\begin{aligned} g(f(x)) &= \frac{-3 \pm \sqrt{1 + 4(x^2+3x+2)}}{2} \\ &= \frac{-3 \pm \sqrt{4x^2+12x+9}}{2} = \frac{-3 \pm (2x+3)}{2} = x \quad \text{or} \quad -x-3. \end{aligned}$$

We get x for the composite value, as expected, but where did the second answer come from?

Let us go back to the definition of inverse. In order for a function to have an inverse from its range to its domain, it must be one-one. This is not the case for the quadratic $f(x)$, where each value exceeding $-1/4$ is assumed twice. In order to provide for an inverse, we restrict the domain of $f(x)$ to a set upon which it is one-one.

Let

$$f_1(x) = f(x) = x^2 + 3x + 2$$

for $x \geq -3/2$. The inverse $g_1(x)$ of $f_1(x)$ is the function

$$g_1(x) = \frac{-3 + \sqrt{1+4x}}{2}$$

defined for $x \geq -1/4$. As x increases from $-1/4$, the value $g_1(x)$ increases from $-3/2$. Note that the graphs of $f_1(x)$ and $g_1(x)$ are images of each other reflected in the line $y = x$. Now

$$g_1(f_1(x)) = \frac{-3 + \sqrt{1 + 4(x^2+3x+2)}}{2} = \frac{-3 + \sqrt{(2x+3)^2}}{2}.$$

Observe that, by definition of the radical, $\sqrt{z}$ refers to the *positive* square root of z. Since $x \geq -3/2$, the positive square root of $(2x+3)^2$ is equal to $2x+3$ and we find that

$$g_1(f_1(x)) = \frac{-3 + (2x - 3)}{2} = x.$$

Now define

$$f_2(x) = f(x) = x^2 + 3x + 2$$

for $x \leq -3/2$. The inverse of *this* function is

$$g_2(x) = \frac{-3 - \sqrt{1+4x}}{2}$$

defined for $x \geq -1/4$. As x increases from $-1/4$ the value of $g_2(x)$ decreases from $-3/2$. Again, observe that the graphs of $f_2(x)$ and $g_2(x)$ are reflections of each other in the line $y = 3$. We have that

$$g_2(f_2(x)) = \frac{-3 - \sqrt{1 + 4(x^2 + 3x + 2}}{2} = \frac{-3 - \sqrt{(2x+3)^2}}{2}.$$

This time, since $x \leq -3/2$, we have that $2x+3 \leq 0$ and the positive square root of $(2x+3)^2$ is $-(2x+3)$. Therefore

$$g_2(f_2(x)) = \frac{-3 - (-(2x+3))}{2} = \frac{-3 + 2x + 3}{2} = x.$$