THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

Sunday, March 10, 2019

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. (a) Determine necessary and sufficient conditions on the sextuple (a, b, c, d, e, f) with $a \leq b \leq c \leq d \leq e \leq f$ in order that there exist four numbers for which a, b, c, d, e, f are the pairwise sums.

   (b) Where the set of four numbers exist as in (a), determine when there is more than one possibility and show that the sum of the squares of the numbers is the same for the different possibiities.

2. For $n = 1, 2, \ldots$, let

$$x_n = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k}.$$

   Prove that $\lim_{n\to\infty} x_n$ exists and find it.

3. The positive integer n is said to be an SP (square-pair) number if the set $\{1, 2, \ldots, 2n\}$ can be partitioned into n pairs such that the sum of the numbers in each pair is a perfect square.

   (a) Prove that n is an SP number for which the sums of the members of the pairs are equal if and only if $n = 2m(m + 1)$ for some positive integer m.

   (b) Show that 10 is not a SP number.

   (c) Prove that there are infinitely many SP numbers that are not of the form $2m(m + 1)$.

4. Determine

$$\int_0^1 \left(4 \arctan x + \pi \tan \frac{\pi x}{4}\right) dx.$$

5. Let $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Determine the minimum value of

$$S = \frac{x_1}{x_2 + x_3 + \cdots + x_n} + \frac{x_2}{x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + x_{n-1}}$$

   (n terms on the right side) and the value of $(x_1, x_2, \ldots, x_n)$ where this minimum is attained.

6. Let m, n be positive integers; let A be a $m \times n$ matrix, B be a $n \times m$ matrix, and BA an invertible $n \times n$ matrix.

   (a) What are the eigenvalues of the matrix $A(BA)^{-1}B$?

   (b) Give an example of this situation when $(m, n) = (3, 2)$.

7. Let $p(z) = z^3 + az^2 + bz + c$ be a cubic polynomial with complex coefficients. Prove that

$$\sup\{|p(z)| : |z| \leq 1\} \geq 1.$$

8. Suppose a and b are elements of a group and r is a positive integer for which $(ab)^r = a$. Prove that $ab = ba$.

9. Suppose that $\{f_n(x) : n = 1, 2, 3, \ldots\}$ is a sequence of differentiable functions defined on an open interval I for which $|f_n'(x)| \leq M$ for some contant M, all $x \in I$ and all positive integers n. Suppose further that $\lim_{n\to\infty} f_n(x) = g(x)$. Prove that $g(x)$ is continuous on I.

10. Let S be the subset of the open interval $(0, 1)$ consisting of those numbers that do not have a terminating binary (base 2) expansion (*i.e.*, are not rationals of the form $r/2^s$ for integers r and s). For $x \in (0, 1)$, let $f_n(x)$ be the number of 0's among the first n digits after the decimal point in the binary expansion of x, divided by n. Let A be the set of numbers x in S for which

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists, and let B be the set of numbers in S for which the limit does not exist.

(a) Prove that both A and B are uncountable.

(b) Does there exist a number in A for which $f(x) = x$?

END

Solutions.

1. (a) Determine necessary and sufficient conditions on the sextuple (a, b, c, d, e, f) with $a \leq b \leq c \leq d \leq e \leq f$ in order that there exist four numbers for which a, b, c, d, e, f are the pairwise sums.

   (b) Where the set of four numbers exist as in (a), determine when there is more than one possibility and show that the sum of the squares of the numbers is the same for the different possibiities.

(a) *Solution.* Suppose that four such numbers, p, q, r, s, exist with $p \leq q \leq r \leq s$. Then $a = p + q$, $b = p + r$, $\{c, d\} = \{p + s, q + r\}$, $e = q + s$ and $f = r + s$. Then $a + f = b + e = c + d = p + q + r + s$.

We show that these conditions are sufficient. Consider the two systems of equations:

$$a = p + q; \qquad b = p + r; \qquad c = q + r;$$

and

$$c = q + r; \qquad e = q + s; \qquad f = r + s.$$

The first system is satisfied by

$$(p, q, r) = \left(\frac{1}{2}(a + b - c), \frac{1}{2}(a - b + c), \frac{1}{2}(-a + b + c)\right)$$

and the second by

$$(q, r, s) = \left(\frac{1}{2}(c + e - f), \frac{1}{2}(c - e + f), \frac{1}{2}(-c + e + f)\right).$$

Since $a + f = b + e$, the two expressions for each of q and r are equal. Therefore, the sextuple consists of the pairwise sums of

$$\left(\frac{1}{2}(a + b - c), \frac{1}{2}(a - b + c), \frac{1}{2}(-a + b + c), \frac{1}{2}(-c + e + f)\right).$$

We can obtain a second solution by replacing c by d so that $d = q + r$.

(b) *Solution 1.* There are two possible quartets when $c \neq d$. Keeping the notation of (a) for the two solutions and letting $u = a + f = b + e = c + d$, we find that

$$3u = a + b + c + d + e + f = 3(p + q + r + s),$$

whence $u = p + q + r + s$. Therefore

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 + e^2 + f^2 &= 3(p^2 + q^2 + r^2 + s^2) + 2(pq + pr + ps + qr + qs + rs) \\ &= 3(p^2 + q^2 + r^2 + s^2) + p(q + r + s) + q(p + r + s) + r(p + q + s) + s(p + q + r) \\ &= 3(p^2 + q^2 + r^2 + s^2) + p(u - p) + q(u - q) + r(u - r) + s(u - s) \\ &= 2(p^2 + q^2 + r^2 + s^2) + (p + q + r + s)u = 2(p^2 + q^2 + r^2 + s^2) + u^2, \end{aligned}$$

so that

$$p^2 + q^2 + r^2 + s^2 = \frac{1}{2}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - u^2).$$

Solution 2. When p, q, r, s are the four numbers, $a = p + q$, $b = p + r$, $e = q + s$, $f = r + s$. There are two possibilities for c and d given by $(c, d) = (p + s, q + r)$ and $(c, d) = (q + r, p + s)$.

For the first possibility,

$$\begin{aligned} 2(p^2 + q^2 + r^2 + s^2) &= (p - q)^2 + (p + q)^2 + (r - s)^2 + (r + s)^2 \\ &= (b - d)^2 + a^2 + (b - c)^2 + f^2, \end{aligned}$$

and, for the second,

$$2(p^2 + q^2 + r^2 + s^2) = (b - c)^2 + a^2 + (b - d)^2 + f^2.$$

For both, $x^2 + y^2 + z^2 + w^2$ has the same value.

For example, the sextuple $(8, 12, 16, 18, 22, 26)$ consists of the pairwise sums of $(2, 6, 10, 16)$ and $(1, 7, 11, 15)$. $\blacksquare$ Both sets of four have square sum equal to $396 = \frac{1}{2}(1948 - 1156)$.

2. For $n = 1, 2, \ldots$, let

$$x_n = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n} \frac{2^k}{k}.$$

Prove that $\lim_{n\to\infty} x_n$ exists and find it.

Solution. The sequence is $\{1, 3/2, 5/3, 5/3, 24/15, 91/60, \ldots\}$. It can be checked that, for each n,

$$x_{n+1} = \frac{n+2}{2(n+1)}(x_n + 1),$$

whereupon

$$x_{n+1} - x_n = \frac{(n+2) - nx_n}{2(n+1)}.$$

Observe that $x_4 > 3/2$. Suppose, as an induction hypothesis, that $x_n > (n + 2)/n$. Then

$$x_{n+1} > \frac{n+2}{2(n+1)}\left[\frac{2n+2}{n}\right] = \frac{n+2}{n} > \frac{n+3}{n+1}.$$

Therefore, for $n \geq 4$, $nx_n > n + 2$ and so $x_{n+1} < x_n$. Thus, $\{x_n\}$ is an eventually decreasing nonnegative sequence and so has a limit. Let this limit be c.

Since

$$\frac{x_{n+1}}{x_n + 1} = \frac{n+2}{2(n+1)},$$

$c/(c + 1) = 1/2$, whence $c = 1$.

[Iranian University Student Mathematics Competition, March, 1980]

3. The positive integer n is said to be an SP (square-pair) number if the set $\{1, 2, \ldots, 2n\}$ can be partitioned into n pairs such that the sum of the numbers in each pair is a perfect square.

(a) Prove that n is an SP number for which the sums of the members of the pairs are equal if and only if $n = 2m(m + 1)$ for some positive integer m.

(b) Show that 10 is not a SP number.

(c) Prove that there are infinitely many SP numbers that are not of the form $2m(m + 1)$.

(a) *Solution 1.* Suppose that n is an SP number and that the sum for each pair is the square s^2.

$$ns^2 = 1 + 2 + \cdots + 2n = n(2n + 1)$$

from which $2n+1 = s^2$. Thus s^2 is odd, and has the form $(2m+1)^2 = 4m^2+4m+1$. Hence $n = 2m(m+1)$.

On the other hand, if $n = 2m(m + 1)$, then the partition

$$((2m + 1)^2 - 1, 1), ((2m + 1)^2 - 2, 2), \ldots (2m^2 + 2m + 1, 2m^2 + 2m)$$

is a partition of the desired type.

Solution 2. Suppose that n is a SP number and we have the pairs $(1,a)$ and $(b,2n)$, which could be the same. If $1+a=b+2n$, then $a-b=2n-1$. Since $b \leq 2n$ and $b \geq 1$, then $a-b \leq 2n-1$ with equality if and only if $a=2n$ and $b=1$. Therefore $1+2n$ is an odd square $(2m+1)^2 = 4m(m+1)+1$, from which we find that $n=2m(m+1)$. The rest is as in the first solution.

(b) *Solution.* Suppose, if possible, there is a partition of $\{1,2,\ldots,20\}$ that fulfils the condition for 10 to be an SP number. The partition must include the pair $(18,7)$. The only possible pairs that include 4 are $(4,5)$ and $(4,12)$. If the set of pairs include $(18,7)$ and $(4,5)$, then $(9,7)$ and $(20,5)$ are ruled out, and so that leaves $(9,16)$ and $(20,16)$. Since both of these pairs cannot be included, we are at an impasse.

On the other hand, if $(18,7)$ and $(4,12)$ are included, then $(9,7)$ and $(13,12)$ are excluded, and we must include $(9,16)$ and $(13,3)$. But then $(20,16)$ is excluded and we must include $(20,5)$. Since $(11,5)$ is now excluded, we must include $(11,14)$. But then $(2,7)$ and $(2,14)$ are excluded, and we cannot have a pair including 2.

(c) *Solution 1.* [I. Bar-Natan; S. Chow] Let $m \geq 2$ and $n = 2m(m+1)-4$. Then $n > 9$ and the set of pairs

$$(1,8),(2,7),(3,6),(4,5),(9,4m(m+1)-8),\ldots,(9+r,4m(m+1)-(8+r)),\ldots,(2m(m+1),2m(m+1)+1)$$

gives the desired partition, where $0 \leq r \leq 2m(m+1)-9$.

Solution 2. [P. Chaitin] Let $n = 2m(3m+1)$ so that $2n = 4m(3m+1) = 12m^2+4m = (4m+1)^2-(2m+1)^2$. Then n is a SP number with pairs $(k,(2m+1)^2-k)$ with $1 \leq k \leq 2m(m+1)$ and $((2m+1)^2+k,(4m+1)^2-(2m+1)^2-k)$ for $0 \leq k \leq 4m^2-1$.

Solution 3. We can start with a partition for some SP number, and augment it to a partition for a larger SP number. Here is the crucial step that can be iterated. Suppose n is a SP number and we have a suitable partition of $\{1,2,\ldots,2n\}$. Pick an odd square $(2r+1)^2$ that exceeds $4n+1$. Then we get a partition for the SP number $2r^2+2r-n$ that consists of the n pairs partitioning $\{1,2,\ldots,2n\}$ along with the $2r^2+2r-2n$ pairs

$$(4r^2+4r-2n,2n+1),(4r^2+4r-2n-1,2n+2),\ldots,(2r^2+2r+1,2r^2+2r).$$

(c) *Solution 4.* Begin with the square m^2 and, where possible, select a square r^2 that satisfies the two conditions (1) $m^2 \leq 2(r^2-1)$ and (2) $3r^2 \leq 2m^2$. Then we have the following partition for $n = m^2-r^2$.

$$(r^2-1,1),(r^2-2,2),\ldots,(m^2-r^2+1,2r^2-m^2-1),$$

$$(2(m^2-r^2),2r^2-m^2),(2(m^2-r^2)-1,2r^2-m^2+1),\ldots,(r^2,m^2-r^2).$$

The conditions on m and r ensure that for each of the two chunks of pairs, the first entries decrease and the second entries increase.

It remains to ensure that for sufficiently large m, we can find r such that

$$\frac{1}{2}(m^2+1) \leq r^2 \leq \frac{2}{3}m^2.$$

But this follows from the fact that the difference between the square roots of the extremes of the inequality is greater than 1 for sufficiently large m and so contains an integer.

(c) *Solution 5.* Inspired by the partition

$$(14,2),(13,3),(12,4),(11,5),(10,6),(9,7),(8,1)$$

for $n=7$, we let n be of the form $2m^2-1$ and consider the partition consisting of the $2m^2-2$ pairs $(4m^2-k,k)$ for $2 \leq k \leq 2m^2-1$, and $(2m^2,1)$ This will be an acceptable partition provided $2m^2+1=r^2$ for some integer r. But this is a Pell's equation and so has infinitely many solutions.

4. Determine

$$\int_0^1 \left(4\arctan x + \pi \tan \frac{\pi x}{4}\right) dx.$$

Solution 1. Let $f(x) = (4/\pi)\arctan x$ and $g(x) = \tan \frac{\pi x}{4}$. Then $f(0) = g(0) = 0$, $f(1) = g(1) = 1$ and f and g are inverses: $f(g(x)) = g(f(x)) = x$ for $x \in [0,1]$. It can be seen from a sketch that

$$\int_0^1 (f(x) + g(x)\, dx = 1.$$

Alternatively, making the substitution $t = g(x)$ whereupon $dt = g'(x)dx = (1/f'(t))dx$, we have

$$\int_0^1 g(x)dx = \int_0^1 tf'(t)dt = [tf(t)]_0^1 - \int_0^1 f(t)\, dt = 1 - \int_0^1 f(x)\, dx,$$

which yields the same result.

The value of the integral in the problem is therefore π.

Solution 2. [R. Chow] Using the respective substitutions $x = \tan\theta$ and $x = (4/\pi)\theta$, we find that

$$\begin{aligned}\int_0^1 4\arctan x\, dx + \int_0^1 \pi \tan \frac{\pi x}{4}\, dx &= 4\left[\int_0^{\pi/4} \theta \sec^2\theta\, d\theta + \int_0^{\pi/4} \tan\theta\, d\theta\right] \\ &= 4\left[\int_0^{\pi/4} (\theta\sec^2\theta + \tan\theta)\, d\theta\right] = 4\left|\theta\tan\theta\right|_0^{\pi/4} = \pi.\end{aligned}$$

Solution 3. Integration by parts yields

$$\int \arctan x dx = x\arctan x - \frac{1}{2}\log(1+x^2),$$

so that

$$\int_0^1 4\arctan x dx = \pi - 2\log 2.$$

Also

$$\int_0^1 \pi\tan\frac{\pi x}{4}\, dx = \left|-4\log\cos\frac{\pi x}{4}\right|_0^1 = -4\log 2^{-1/2} = 2\log 2.$$

Hence the desired answer is π.

5. Let $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Determine the minimum value of

$$S = \frac{x_1}{x_2 + x_3 + \cdots + x_n} + \frac{x_2}{x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + x_{n-1}}$$

(n terms on the right side) and the value of $(x_1, x_2, \ldots, x_n)$ where this minimum is attained.

Solution 1. Let $s = x_1 + x_2 + \cdots + x_n$ and, for each i, let $u_i = x_1 + \cdots + \hat{x}_i + \cdots + x_n$ be the sum with the term x_i left out. Note that $\sum_{i=1}^n u_i = (n-1)s$.

$$\begin{aligned}S &= \sum_{i=1}^n \frac{s - u_i}{u_i} = \left[s\sum_{i=1}^n \frac{1}{u_i}\right] - n \\ &= \frac{1}{n-1}\left(\sum_{i=1}^n u_i\right)\left(\sum_{i=1}^n \frac{1}{u_i}\right) - n \\ &\geq \frac{n^2}{n-1} - n = \frac{n}{n-1},\end{aligned}$$

by the Cauchy-Schwarz inequality, with equality if and only if all the u_i are equal, if and only if all the x_i are equal.

Solution 2. [S. Li] Wolog, let $x_1 + x_2 + \cdots + x_n = 1$, so that

$$S = \sum_{i=1}^{n} \frac{x_i}{1 - x_i} = \left(\sum_{i=1}^{n} \frac{1}{1 - x_i} \right) - n.$$

By the harmonic-geometric mens inequality applied to the $n-$tple $\{(1 - x_i)\}$, we find that

$$\frac{n}{\sum_{i=1}^{n} (1 - x_i)^{-1}} \leq \frac{1}{n} \sum_{i=1}^{n} (1 - x_i) = \frac{n-1}{n},$$

with equality if and only if all x_i are equal.

Hence

$$S \geq \frac{n^2}{n-1} - n = \frac{n}{n-1}$$

with equality if and only if all the x_i sre equal.

Solution 3. [J. Guo] Recall the Rearrangement Inequality: *if* $0 \leq u_1 \leq u_2 \leq \cdots \leq x_n$ *and* $0 \leq v_1 \leq v_2 \leq \cdots + \leq v_n$*, then*

$$\sum_{i=1}^{n} u_{\sigma(i)} v_i \leq \sum_{i=1}^{n} u_i v_i,$$

whenever σ *is apermutation on* $\{1, 2, \ldots, n\}$*. Equality occurs if and only if* σ *is the identity permutation (or in the case some of the* u_i *are equal, a modification of the identity that switches the equal* u_i*).*

Wolog, suppose that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and, for $1 \leq k \leq n$, let

$$d_k = x_1 + x_2 + \cdots + \hat{x_k} + \cdots + x_n = \left(\sum_{i=1}^{n} x_i \right) - x_k.$$

Since $1/d_1 \leq 1/d_2 \leq \cdots \leq 1/d_n$,

$$\sum_{i=1}^{n} \frac{x_{\sigma(i)}}{d_i} \leq \sum_{i=1} n \frac{x_i}{d_i} = s$$

for each permutation σ. For $1 \leq j \leq n-1$, let σ_j be the cyclic permutation defined by

$$\sigma(i) = \begin{cases} i + j, & \text{if } 1 \leq i \leq n - j; \\ (i + j) - n, & \text{if } n - j < i \leq n. \end{cases}$$

Then

$$\sum_{j=1}^{n-1} \sum_{i=1}^{n} \frac{x_{\sigma_j(i)}}{d_i} \leq (n-1)s.$$

But the left side is equal to

$$\sum_{i=1}^{n} \frac{1}{d_i} (x_{i+1} + x_{i+2} + \cdots + +x_n + x_1 + \cdots + x_{i-1}) = \sum_{i=1}^{n} \frac{d_i}{d_i} = n.$$

The result follows and $n/(n-1)$ is an upper bound for S.

This upper bound is achieved when all the x_i are equal. If the x_i are not all equal, then some cyclic permutation σ will give a nonincreasing sequence $x_{\sigma(i)}$ and we will obtain a strict inequality in one of the sums added together for the final result. Thus equality holds if and only if all x_i are equal.

Solution 4. Fix $t = x_1 + x_2 + \cdots + x_n$ and let S be defined on the compact set K defined by $x_i \geq 0$ for $1 \leq i \leq n$ and $x_1 + x_2 + \cdots + x_n = t$. Then, on K, S must assume its minimum value.

We show that, if not all the x_i are equal, then S must assume a lower value somewhere else on K. Wolog, suppose that $u = x_1 \neq x_2 = v$. Let $2w = x_1 + x_2$. Observe that, by the inequality of the harmonic, arithmetic and geometric means inequality,

$$\frac{1}{2}(u^2 + v^2) > w^2 > uv.$$

Let $p = x_3 + \cdots + x_n$ and $q = \sum_{r=3}^{n} x_r(t - x_r)^{-1}$. Observe that $w + w + p = u + v + p = x_1 + x_2 + \cdots + x_n = t$.

Then

$$\begin{aligned} S(u, v, x_3, \ldots, x_n) &= \left[\frac{u}{p+v} + \frac{v}{p+u}\right] + q \\ &= \left[\frac{u^2 + v^2 + p(u+v)}{uv + p(u+v) + p^2}\right] + q \\ &> \left[\frac{2w^2 + 2pw}{w^2 + 2pw + p^2}\right] + q = \frac{2w}{p+w} + q = s(w, w, x_3, \ldots, x_n). \end{aligned}$$

Thus, S can attains its minimum value $n(n-1)^{-1}$ only when all the x_i are equal.

Solution 5. Wolog, suppose $x_1 + x_2 + \cdots + x_n = 1$. Then $S = \sum_{i=1}^{n} \frac{x_i}{1 - x_i}$. Since the function $f(t) = t/(1-t)$ has positive first and second derivatives on $(0, 1)$ it is increasing and convex. Hence

$$S/n = (1/n)(f(x_1) + f(x_2) + \cdots + f(x_n)) \geq f(1/n) = 1/(n-1),$$

so that $S \geq n/(n-1)$ with equality if and only if all the x_i are equal.

6. Let m, n be positive integers; let A be a $m \times n$ matrix, B be a $n \times m$ matrix, and BA an invertible $n \times n$ matrix.

   (a) What are the eigenvalues of the matrix $A(BA)^{-1}B$?

   (b) Give an example of this situation when $(m, n) = (3, 2)$.

(a) *Solution 1.* [based on Y. Zhong & J. Guo] Since BA is invertible, its rank is equal to n. Therefore the rank of A is at least n. But the rank of A does not exceed the minimum of m and n, so it follows that $m \geq n$.

Observe that

$$(A(BA)^{-1}B)(A(BA)^{-1}B) = A(BA)^{-1}(BA)(BA)^{-1}B = A(BA)^{-1}B,$$

so that the minimum polynomial of $A(BA)^{-1})B$ is $t^2 - t = t(t-1)$. Therefore the only possible eigenvalues are 0 and 1.

We first note that if $P = (p_{ij})$ is a $m \times n$ matrix and $Q = (q_{ij})$ is a $n \times m$ matrix, then PQ is an $m \times m$ and QP is an $n \times n$ matrix and their traces are equal. Note that both traces are equal to

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} q_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{m} q_{ji} p_{ij}.$$

Applying this we see that the traces of $A[(BA)^{-1}B]$ and $[(BA)^{-1}B]A = (BA)^{-1}BA = I_n$ are equal. Since the trace of I_n is equal to n as well as the sum of its eigenvalues, The trace of $A(BA)^{-1}B$ is equal to the trace of $(BA)^{-1}BA = I_n$, namely n.

Hence 1 is an eigenvalue with multiplicity n and 0 an eigenvalue with multiplicity $m - n$.

Solution 2. We first note that when $m < n$, BA must always be singular, so that the situation is not possible. We can write $A = (c_1, c_2, \ldots, c_n)$ where each c_i is a column $m-$vector, and $B = (r_1, r_2, \ldots, r_n)^{\mathbf{t}}$ where each r_i is a row $m-$vector. The set $\{r_i\}$ is linearly dependent, so there exist constants α_i not all zeros such that $\sum \alpha\alpha_i r_i = 0$.

Since the ith row of BA is $(r_i c_1, r_i c_2, \ldots, r_i c_n)$ and there is a vanishing nontrivial linear combination with coefficients α_i of the rows which vanishes. Therefore BA must be singular. Henceforth, we assume that $m \geq n$.

Let $m = n$. Then A and B are square matrices of the same order, and $A(BA)^{-1}B = AA^{-1}B^{-1}B = I$; thus the only eigenvalue is 1.

Let $m > n$. Suppose that u is a nonzero $m-$vector for which

$$A(BA)^{-1}Bu = \lambda u.$$

Then, multiplying on the left by B, we obtain

$$Bu = \lambda Bu.$$

Then, either $\lambda = 1$ or $Bu = O$. In the latter case, $A(BA)^{-1}Bu = O$ so that the eigenvalue corresponding to u is 0.

Since BA is invertible, A is not the zero matrix, and so there is a $n-$vector v for which $Av \neq O$. In this case

$$A(BA)^{-1}B(Av) = A(BA)^{-1}(BA)v = Av,$$

so that Av has eigenvalue 1. Since $m < n$, the columns of B are linearly dependent, and so the columns of $[A(BA)^{-1}]B$ are linearly dependent with the same coefficients. Hence $\det A(BA)^{-1}B = 0$ and 0 is an eigenvalue.

(b)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$$(BA)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$A(BA)^{-1}B = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

The vector $(-1, 0, 1)^{\mathbf{t}}$ is an eigenvector with eigenvalue 0, while $(1, 0, 1)^{\mathbf{t}}$ and $(0, 1, 1)^{\mathbf{t}}$ are eigenvectors with eigenvalue 1.

7. Let $p(z) = z^3 + az^2 + bz + c$ be a cubic polynomial with complex coefficients. Prove that

$$\sup\{|p(z)| : |z| \le 1\} \ge 1.$$

Solution 1. Let $\omega = \frac{1}{2}(1 + i\sqrt{3})$ be an imaginary cube root of unity, and let

$$f(z) = \frac{1}{3}[p(z) + p(\omega z) + p(\omega^2 z)].$$

Noting that $1 + \omega + \omega^2 = 0$, we find that $f(z) = z^3 + c$.

Let v be a cube root of $c/|c|$, so that $|v| = 1$. Then

$$f(v) = \frac{c}{|c|} + c = c\left(\frac{1}{|c|} + 1\right)$$

and

$$|f(v)| = |c|\left(\frac{1}{|c|} + 1\right) = 1 + |c| \ge 1.$$

Since

$$3 \le 3|f(v)| = |p(v) + p(\omega v) + p(\omega^2 v)| \le |p(v)| + |p(\omega v)| + |p(\omega^2 v)|.$$

Thus, at least one of $p(v)$, $p(\omega v)$ and $p(\omega^2 v)$ has absolute value not less than 1 and the result follows.

Note. We actually have a stronger inequality.

Solution 2. [S. Li] By Rouché's Theorem, if f and g are two complex functions for which $|g(z)| < |f(z)|$ for $|z| = 1$, then $f(z)$ and $f(z) - g(z)$ have the same number of zeros inside the closed unit disc. Suppose, if possible, that $\sup\{|p(z)| : |z| \le 1\} < 1$. Then, applying Rouché's Theorem to $f(z) = z^3$ and $g(z) = p(z)$, the functions z^3 and $az^2 + bz + c$ would have the same number of roots (counting multiplicity) in the closed unit disc. But this is impossible, since the latter function is a quadratic. Hence the result holds.

Solution 3. [I. Bar-Natan] Let $q(z) = p(z)/z^4$. Then

$$\begin{aligned}\oint_{|z|=1} q(z)\,dz &= \oint_{|z|=1} \frac{1}{z}\,dz + \oint_{|z|=1} \frac{a}{z^2}\,dz + \oint_{|z|=1} \frac{b}{z^3}\,dz + \oint_{|z|=1} \frac{c}{z^3}\,dz \\ &= 2\pi i + 0 + 0 + 0 = 2\pi i.\end{aligned}$$

Therefore

$$2\pi \le \oint_{|z|=1} |q(z)|\,dz = \oint_{|z|=1} |p(z)|\,dz \le 2\pi \sup_{|z|=1} |p(z)|,$$

from which the result follows.

Note. This result can be extended to monic polynomials of arbitrary degree n. All three arguments work, where in the first, you work with the value of the polynomial at the nth roots of unity.

8. Suppose a and b are elements of a group and r is a positive integer for which $(ab)^r = a$. Prove that $ab = ba$.

Solution 1. [S. Li]

$$ba = b(ab)^r = (a^{-1}a)b(ab)^r = a^{-1}(ab)^r(ab) = a^{-1}a(ab) = ab.$$

Solution 2. [B. Fattin]

$$\begin{aligned}(ab)^{2r-1}(ab) &= (ab)^{2r} = (ab)^r(ab)^r = (ab)^r a \\ &= (ab)^{r-1}a(ba) = (ab)^{r-1}(ab)^r(ba) = (ab)^{2r-1}(ba),\end{aligned}$$

Multiplying on the left by the inverse of $(ab)^{2r-1}$, we find that $ab = ba$.

Solution 3. Suppose $(ab)^r = a$. Then $ab(ab)^{r-1} = a$ implies that $b(ab)^{r-1} = e = (ba)^{r-1}b$. Hence $(ab)^{r-1} = b^{-1} = (ba)^{r-1}$. Therefore $(ba)^{r-2}b = a^{-1}b^{-1}$ and $(ba)^{r-2}b = b^{-1}a^{-1}$.

Therefore $a^{-1}b^{-1} = b^{-1}a^{-1}$ and so

$$ab = ab(a^{-1}b^{-1})ba = ab(b^{-1}a^{-1})ba = ba.$$

9. Suppose that $\{f_n(x) : n = 1, 2, 3, \ldots\}$ is a sequence of differentiable functions defined on an open interval I for which $|f_n'(x)| \leq M$ for some constant M, all $x \in I$ and all positive integers n. Suppose further that $\lim_{n\to\infty} f_n(x) = g(x)$. Prove that $g(x)$ is continuous on I.

Solution. Let u and v be any two elements of I with $u < v$. Let $\epsilon > 0$. Choose N such that, for $n \geq N$, $|f_n(u) - g(u)| < \epsilon$ and $|f_n(v) - g(v)| < \epsilon$ for $n \geq N$. Then, for $n \geq N$ and a suitable $w \in (u, v)$, we have that

$$\begin{aligned} |g(v) - g(u)| &\leq |g(v) - f_n(v)| + |f_n(v) - f_n(u)| + |f_n(u) - g(u)| \\ &\leq 2\epsilon + |f_n(v) - f_n(u)| = 2\epsilon + |f_n'(w)|(v - u) \\ &\leq 2\epsilon + M(v - u). \end{aligned}$$

Since this holds for all positive ϵ, $|g(v) - g(u)| \leq M|v - u|$ for all u and v in I, and the result follows.

10. Let S be the subset of the open interval $(0, 1)$ consisting of those numbers that do not have a terminating binary (base 2) expansion (*i.e.*, are not rationals of the form $r/2^s$ for integers r and s). For $x \in (0, 1)$, let $f_n(x)$ be the number of 0's among the first n digits after the decimal point in the binary expansion of x, divided by n. Let A be the set of numbers x in S for which

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists, and let B be the set of numbers in S for which the limit does not exist.

(a) Prove that both A and B are uncountable.

(b) Does there exist a number in A for which $f(x) = x$?

Solution. (a) For each nonrational number $x \in (0, 1)$ with binary expansion $0.a_1a_2...$, define $g(x) = 0.b_1b_2\ldots$ where $b_{2n-1}b_{2n} = 01$ when $a_n = 0$ and $b_{2n-1}b_{2n} = 10$ when $a_n = 1$. The domain of g is uncountable, g is one-one so its image is uncountable, and $f_{2n}(g(x)) = 1/2$. It follows that $f(g(x)) = 1/2$. Therefore A is uncountable.

We begin with a basic number b in $(0, 1)$ for which the limit does not exist amd then create uncoutably many modifications. Let

$$b = 0.011000011111111000000000000000011111111111111111111111111111111110\ldots,$$

where the digits come in consecutive tranches T_r $(r \geq 1)$, where T_r has 2^{r-1} digits. When r is odd, these digits are all zero. When r is even, these digits are all one. Let p_r be the ratio of zeros to the number of digits in the first r tranches, so that $p_1 = 1$, $p_2 = 1/3$, $p_3 = 5/7$, $p_4 = 5/15 = 1/3$, $p_5 = 21/31$. In general, for each positive integer s,

$$p_{2s-1} = \frac{1 + 4 + \cdots + 4^{s-1}}{2^{2s-1} - 1} > \frac{2^{2s-2}}{2^{2s-1} - 1} > \frac{1}{2},$$

and

$$p_{2s} = \frac{1 + 4 + \cdots + 4^{s-1}}{1 + 2 + 2^2 + \cdots + 2^{2s-1}} = \frac{1 + 4 + \cdots + 4^{s-1}}{(1 + 2)(1 + 2^2 + \cdots + 2^{2(s-1)})} = \frac{1}{3}.$$

Thus,

$$\limsup_{s\to\infty} p^{2s-1} \geq \frac{1}{2}$$

and

$$\lim_{s\to\infty} p_{2s} = \frac{1}{3}.$$

Let x be any nonrational number in $(0,1)$ so that it has a nonterminating binary expansion. We modify b to obtain a number b_x by inserting at the right end of the rth tranche of b the rth digit of b. In b_x the ratio of the number of zeros to the number of digits in the first r tranches (where $r = 2s-1$ or $r = 2s$) lies in the interval

$$\left[\frac{1+4+\cdots+4^{s-1}}{2^r-1+r}, \frac{1+4+\cdots+4^{s-1}+r}{2^r-1+r}\right].$$

This interval has length less than $r/(2^{r-1}-1)$ and contains p_r.

When $t = 2^r - 1 + r$,

$$p_r - \frac{r}{2^r-1} < f_t(b_x) < p_r + \frac{r}{2^r-1},$$

so that $\limsup_{n\to\infty} f_n(b_x) \geq 1/2$ and $\liminf_{n\to\infty} f_n(b_x) \leq 1/3$.

(b) There is such a number. Let $u_1 = 0 = 0.0$, $v_1 = 0$, $u_2 = 0.01$, $v_2 = 1/2$. We define by induction sequences $\{u_n\}$ and $\{v_n\}$ such that each u_n has a terminating binary expansion with n digits after the decimal point (the last of which can be 0), u_{n+1} has u_n as a prefix and v_n is the number of 0's in the expansion (after the decimal point) of u_n divided by n.

Let $k \geq 2$. If $v_k > u_k$, then append 1 to the expansion of u_k to get $u_{k+1} > u_k$; if $v_k \leq u_k$, append 0 to the expansion of u_k to get $u_{k+1} = u_k$. Let v_{k+1} be the number of 0's in the expansion of u_{k+1} divided by $k+1$. Observe that when $v_k > u_k$, then $v_{k+1} < v_k$, while when $v_k < u_k$, then $v_{k+1} > v_k$.

Since either adding 0's or adding 1's indefinitely sends v_n towards 0 and 1 respectively, and $v_n - u_n$ cannot be eventually either all positive or all negative, we must switch between appending 0 and appending 1 indefinitely. Let $x = \lim u_n$. We have to show that $x = \lim_{n\to\infty} v_n = \lim_{n\to\infty} f_n(x)$.

Let $\epsilon > 0$ be given and choose N such that $1/N < \epsilon$ and $v_N \leq u_N = u_{N+1} < v_{N+1}$. Then

$$x - \epsilon < x - \frac{1}{2^N} < u_{N+1} < v_{N+1} < u_N + \frac{1}{2^N} < x + \epsilon.$$

Suppose that $n \geq N$ and it has been shown that $x-\epsilon < v_n < x+\epsilon$. If $v_{n+1} > v_n$, then $v_{n+1} = (a+1)/(n+1)$ and $v_n = a/n$ for some positive integer $a < n$, so that $v_{m+1} - v_m = (m-a)/m(m+1) \geq 1/(m+1)$. Hence

$$x - \epsilon < v_n < v_{n+1} < v_n + \frac{1}{n+1} \leq u_n + \frac{1}{n+1} < x + \frac{1}{n} < x + \epsilon.$$

On the other hand, if $v_{n+1} < v_n$, then

$$x + \epsilon > v_n > v_{n+1} > v_n - \frac{1}{n+1} > u_n - \frac{1}{n+1} > x - \frac{1}{2^n} - \frac{1}{m+1} > x - \frac{1}{n} > x - \epsilon.$$

Thus, for all $n \geq N$, $|x - v_n| < \epsilon$.

[Note that $(u_n, v_n) = (0.0, 0)$, $(0.01 = 1/4, 1/2)$, $(0.011 = 5/8, 1/3)$, $(0.0110 = 5/8, 1/2)$, $(0.01100 = 5/8, 3/5)$, $(0.011000 = 5/8, 2/3)$, $(0.0110001 = 81/128, 4/70$, $(0.01100011 = 163/256, 1/2)$, $\cdots$.]