THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March 8, 2009

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

It is not necessary to do all the problems. Complete solutions to fewer problems are preferred to partial solutions to many.

1. Determine the supremum and the infimum of

$$\frac{(x-1)^{x-1}x^x}{(x-(1/2))^{2x-1}}$$

for $x > 1$.

2. Let n and k be integers with $n \geq 0$ and $k \geq 1$. Let $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_n$ be $n+1$ distinct points in $\mathbf{R}^k$ and let $y_0, y_1, \cdots, y_n$ be $n+1$ real numbers (not necessarily distinct). Prove that there exists a polynomial p of degree at most n in the coordinates of $\mathbf{x}$ with respect to the standard basis for which $p(\mathbf{x}_i) = y_i$ for $0 \leq i \leq n$.

3. For each positive integer n, let $p(n)$ be the product of all positive integral divisors of n. Is it possible to find two distinct positive integers m and n for which $p(m) = p(n)$?

4. Let $\{a_n\}$ be a real sequence for which

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

converges. Prove that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0 \ .$$

5. Find a 3×3 matrix A with elements in $\mathbf{Z}_2$ for which $A^7 = I$ and $A \neq I$. (Here, I is the identity matrix and $\mathbf{Z}_2$ is the field of two elements 0 and 1 where addition and multiplication are defined modulo 2.)

6. Determine all solutions in nonnegative integers (x, y, z, w) to the equation

$$2^x 3^y - 5^z 7^w = 1 \ .$$

7. Let $n \geq 2$. Minimize $a_1 + a_2 + \cdots + a_n$ subject to the constraints $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 1$. (When $n = 2$, the latter condition is $a_1a_2 = 1$; when $n \geq 3$, the sum on the left has exactly n terms.)

8. Let a, b, c be members of a real inner-product space $(V, \langle , \rangle)$ whose norm is given by $\|x\|^2 = \langle x, x\rangle$. (You may assume that V is $\mathbf{R}^n$ if you wish. Prove that

$$\|a+b\| + \|b+c\| + \|c+a\| \leq \|a\| + \|b\| + \|c\| + \|a+b+c\|$$

for $a, b, c, \in V$.

Please turn over the page for remaining questions.

9. Let p be a prime congruent to 1 modulo 4. For each real number x, let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Determine

$$\sum \left\{ \left\{ \frac{k^2}{p} \right\} : 1 \leq k \leq \frac{1}{2}(p-1) \right\} .$$

10. Suppose that a path on a $m \times n$ grid consisting of the lattice points $\{(x, y) : 1 \leq x \leq m, 1 \leq y \leq n\}$ (x and y both integers) consisting of $mn - 1$ unit segments begins at the point $(1, 1)$, passes through each point of the grid exactly once, does not intersect itself and finishes at the point (m, n). Show that the path partitions the rectangle bounded by the lines $x = 1$, $x = m$, $y = 1$, $y = n$ into two subsets of equal area, the first consisting of regions opening to the left or up, and the second consisting of regions opening to the right or down.

END

1. Determine the supremum and the infimum of

$$\frac{(x-1)^{x-1}x^x}{(x-(1/2))^{2x-1}}$$

for $x > 1$.

Solution. Let $g(x)$ be the function in question and let

$$\begin{aligned} f(x) &= \log g(x) \\ &= (x-1)\log(x-1) + x\log x - (2x-1)\log((2x-1)/2) \\ &= (x-1)\log(x-1) + x\log x - (2x-1)\log(2x-1) + (2x-1)\log 2 \ . \end{aligned}$$

Then

$$\begin{aligned} f'(x) &= \log(x-1) + 1 + \log x + 1 - 2\log(2x-1) - 2 + 2\log 2 \\ &= \log\left[\frac{4x(x-1)}{(2x-1)^2}\right] = \log\left[1 - \frac{1}{(2x-1)^2}\right] \\ &< \log 1 = 0 \ . \end{aligned}$$

Therefore, $f(x)$, and hence $g(x)$ is a decreasing function on $(0, \infty)$.

It is straightforward to check that $\lim_{x\downarrow 1} f(x) = \log 2$, so that $\lim_{x\downarrow 1} g(x) = 2$.

To check behaviour for large values of x, let

$$\begin{aligned} h(u) &= g(u + (1/2)) \\ &= \frac{(u-(1/2))^{u-(1/2)}(u+(1/2))^{u+(1/2)}}{u^{2u}} \\ &= \left(1 - \frac{1}{2u}\right)^u \left(1 + \frac{1}{2u}\right)^u \left(1 - \frac{1}{2u}\right)^{-1/2} \left(1 + \frac{1}{2u}\right)^{1/2} \\ &= \left(1 - \frac{1}{4u^2}\right)^u \left(\frac{1 + (1/2u)}{1 - (1/2u)}\right)^{1/2} \ . \end{aligned}$$

When $v = 1/(2u)$, the logarithm of the first factor is

$$\frac{\log(1-v^2)}{2v} \ ,$$

and an application of l'Hôpital's Rule yields that its limit is 0 as $v \to 0$. It follows that

$$\lim_{u\to\infty} h(u) = 1 \ .$$

Therefore, the desired supremum is 2 and infimum is 1.

2. Let n and k be integers with $n \geq 0$ and $k \geq 1$. Let $\mathbf{x}_0$, $\mathbf{x}_1$, $\cdots$, $\mathbf{x}_n$ be $n+1$ distinct points in $\mathbf{R}^k$ and let y_0, y_1, $\cdots$, y_n be $n+1$ real numbers (not necessarily distinct). Prove that there exists a polynomial p of degree at most n in the coordinates of $\mathbf{x}$ for which $p(\mathbf{x}_i) = y_i$ for $0 \leq i \leq n$.

Solution. For $0 \leq i < j \leq n$, be H_{ij} be the hyperplane $\{\mathbf{z} : (\mathbf{x}_i - \mathbf{x}_j) \cdot \mathbf{z} = 0\}$. Since there are finitely many such hyperplanes, their union is not all of $\mathbf{R}^k$. Therefore, there exists a vector $\mathbf{u}$ for which $(\mathbf{x}_i - \mathbf{x}_j) \cdot \mathbf{u} \neq 0$ for all distinct i, j. Therefore, the real numbers $t_i = \mathbf{x_i} \cdot \mathbf{u}$ are all distinct $0 \leq i \leq n$. There is a real polynomial q of degree at most n for which $q(t_i) = y_i$ $(0 \le i \le n)$. The polynomial $p(\mathbf{x}) = q(\mathbf{x} \cdot \mathbf{u})$ has the desired property.

3. For each positive integer n, let $p(n)$ be the product of all positive integral divisors of n. Is it possible to find two distinct positive integers m and n for which $p(m) = p(n)$?

Solution. No; the function $p(n)$ is one-one. Observe that, since each divisor d can be paired with n/d (which are distinct except when n is square and d is its square root), $p(n) = n^{\tau(n)/2}$, where $\tau(n)$ is the number of divisors of n.

Suppose that $p(m) = p(n)$ and that q is any prime. Let the highest powers of q dividing m and n be q^u and q^v respectively. The exponent of the highest power of q that divides $p(m)$ is $(u\tau(m))/2$ and the highest power of q that divides $p(n)$ has exponent $(v\tau(n))/2$.

Since $p(m) = p(n)$, it follows that $u\tau(m) = v\tau(n)$, so that $u/v = \tau(m)/\tau(n)$. Therefore, for each prime q, u/v is always less than, always greater than or always equal to 1. But the first two options cannot hold (consider any prime that divides neither m nor n), so $u = v$ for each prime q. But then this means that $m = n$.

4. Let $\{a_n\}$ be a real sequence for which

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

converges. Prove that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0 \ .$$

Solution. Recall Abel's Partial Summation Formula:

$$\sum_{k=1}^{n} x_k y_k = (x_1 + x_2 + \cdots + x_n) y_{n+1} + \sum_{k=1}^{n} (x_1 + x_2 + \cdots + x_k)(y_k - y_{k+1}) \ .$$

Applying this to $x_n = a_n/n$ and $y_n = n$. we obtain that

$$a_1 + a_2 + \cdots + a_n = (x_1 + x_2 + \cdots + x_n)(n+1) - \sum_{k=1}^{n} (x_1 + x_2 + \cdots + x_k) \ ,$$

whence

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = \left(\sum_{k=1}^{n} x_k\right)\left(1 + \frac{1}{n}\right) - \frac{1}{n}\sum_{k=1}^{n} (x_1 + x_2 + \cdots + x_k) \ .$$

Let

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{a_k}{k}\right) \ .$$

Then

$$L = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\left(\sum_{k=1}^{n} x_k\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (x_1 + x_2 + \cdots + x_k) \ .$$

To see the latter equality, suppose that $u_n = x_1 + x_2 + \cdots + x_n$ and $v_n = (1/n)(u_1 + u_2 + \cdots + u_n)$, so that

$$v_n - u_n = \frac{1}{n}[(u_1 - u_n) + (u_2 - u_n) + \cdots (u_n - u_n)] \ .$$

Let $\epsilon > 0$ be given. First select n_1 so that $|u_k - u_n| < \frac{1}{2}\epsilon$ for $n \geq n_1$. Since $\lim u_n = L$, there exists a positive integer M for which $|u_n| < M$ for each n. Select $n_2 > n_1$ such that, for all $n > n_2$, $2Mn_1/n < \epsilon/2$. Then, for all $n > n_2$,

$$\begin{aligned} |v_n - u_n| &\leq \sum_{k=1}^{n_1} |v_k - u_k| + \sum_{k=n_1+1}^{n} |v_k - u_k| \\ &\leq \frac{2Mn_1}{n} + \frac{n-n_1}{n}\left(\frac{\epsilon}{2}\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ . \end{aligned}$$

It follows that $\lim v_n = L$, and the desired result obtains.

5. Find a 3×3 matrix A with elements in $\mathbf{Z}_2$ for which $A^7 = I$ and $A \neq I$. (Here, I is the identity matrix and $\mathbf{Z}_2$ is the field of two elements 0 and 1 where addition and multiplication are defined modulo 2.)

Solution. The minimum polynomial of A has degree at most 3 and divides the polynomial

$$t^7 - 1 = (t-1)(t^6 + t^5 + t^4 + t^3 + t^2 + t + 1) \ .$$

Since A is not the identity matrix, its minimum polynomial divides the latter factor. This minimum polynomial cannot be any of the irreducibles t, $t+1$ (by the Factor Theorem) and $t^2 + t + 1$, so it must be one of the irreducibles $t^3 + t^2 + 1$, $t^2 + t + 1$. Indeed

$$t^6 + t^5 + t^4 + t^3 + t^2 + t + 1 = (t^3 + t + 1)(t^3 + t^2 + 1) \ ,$$

over $\mathbf{Z}_2$.

We let A be the companion matrix of the first factor on the right, namely

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \ .$$

Indeed, it can be checked that $A^3 = I + A$, whence $A^6 = I + A^2$ and $A^7 = A + A^3 = A + I + A = I$, as desired.

6. Determine all solutions in nonnegative integers (x, y, z, w) to the equation

$$2^x 3^y - 5^z 7^w = 1 \ .$$

Solution. By parity considerations, we must have that $x \geq 1$. Suppose, first of all, that $y \geq 1$, $z \geq 1$ and $w \geq 1$. Since $2^x 3^y \equiv 1 \pmod 5$, it follows that $x \equiv y \pmod 4$, so that x and y have the same parity. Since $2^x 3^y \equiv 1 \pmod 7$, it follows that $(x, y) \equiv (0,0), (1,4), (2,2), (3,0), (4,4), (5,2) \pmod 6$. Therefore, x and y are both even. Let $x = 2u$ and $y = 2v$. Then

$$5^z 7^w = 2^{2u} 3^{2v} - 1 = (2^u 3^v - 1)(2^u 3^v + 1) \ .$$

The two factors on the right must be coprime (being consecutive odd numbers), so that one of them is a power of 5 and the other is a power of 7. Thus $2^u 3^v \pm 1 = 5^z$ and $2^u 3^v \mp 1 = 7^w$, whence

$$7^w - 5^z = \mp 2 \ .$$

Observe that 7^w is congruent, modulo 25, to one of 1, 7, -1, -7, so that $7^w \mp 2$ is never divisible by 25. Therefore, $z = 1$ and so $w = 1$ and we obtain the solution $(x, y, z, w) = (2, 2, 1, 1)$.

Now we consider the cases where at least one of y, z, w vanishes.

Suppose that $y = 0$. Then $2^x - 1 = 5^z 7^w$. If x is even, then the left side is divisible by 3. Therefore x is odd. But if x is odd, then the left side is not divisible by 5. Therefore $z = 0$ and $2^x - 7^w = 1$. This implies either that $w = 0$ or that x is divisible by 3. There are two immediate possibilities: $(x, w) = (1, 0)$ and $(x, w) = (3, 1)$. Suppose that $x = 3r$ where $r \geq 2$. Then

$$7^w = (2^r)^3 - 1 = (2^r - 1)(2^{2r} + 2^r + 1) ,$$

so that both factors on the right are nontrivial powers of 7. But it is not possible for both factors of the right side to be divisible by 7, and we obtain no further solutions.

Therefore, when $y = 0$, we have only the solutions $(x, y, z, w) = (1, 0, 0, 0), (3, 0, 0, 1)$.

It is not possible for x and y to both exceed 0, while $z = 0$ since $1 + 7^w \equiv 2 \pmod 3$, for each integer w.

Finally, let x and y both exceed 0 and let $w = 0$. Since $5^z + 1 \equiv 2 \pmod 4$, we must have that $x = 1$. Observe that $5^z + 1 \equiv 0 \pmod 9$ only when $z \equiv 3 \pmod 6$. However, when $z \equiv 3 \pmod 6$, $5^z + 1 \equiv 2 \pmod 7$. Therefore, either $5^z + 1$ is not divisible by 9, or it is divisible by 7. In either case, it cannot be of the form $2^x 3^y$ when $y \geq 2$. Therefore, $y = 1$. This leads to the sole solution $(x, y, z, w) = (1, 1, 1, 0)$.

In all, there are four integral solutions to the equation, namely

$$(x, y, z, w) = (2, 2, 1, 1), (1, 1, 1, 0), (3, 0, 0, 1), (1, 0, 0, 0) .$$

7. Let $n \geq 2$. Minimize $a_1 + a_2 + \cdots + a_n$ subject to the constraints $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n + a_n a_1 = 1$. (When $n = 2$, the latter condition is $a_1 a_2 = 1$.)

Solution. If $n \geq 3$ and $a_i = 1/\sqrt{n}$ for each i, then $a_1 + a_2 + \cdots + a_n = \sqrt{n}$. If $a_1 = a_2 = \cdots = a_{n-2} = 0$ and $a_{n-1} = a_n = 1$, $a_1 + a_2 + \cdots + a_n = 2$. Therefore,when $n \geq 3$, the minimum does not exceed the lesser of $\sqrt{n}$ and 2.

Let $n = 2$. By the arithmetic-geometric means inequality, we have that

$$a_1 + a_2 \geq 2\sqrt{a_1 a_2}$$

so that the minimum is 2.

Let $n = 3$. Then

$$\begin{aligned}(a_1 + a_2 + a_3)^2 &= a_1^2 + a_2^2 + a_3^2 + 2(a_1 a_2 + a_2 a_3 + a_3 a_1) \\ &= \frac{1}{2}(a_1^2 + a_2^2) + \frac{1}{2}(a_2^2 + a_3^2) + \frac{1}{2}(a_3^2 + a_1^2) + 2 \\ &\geq a_1 a_2 + a_2 a_3 + a_3 a_1 + 2 = 3 ,\end{aligned}$$

so that the minimum is $\sqrt{3}$.

Let $n = 4$. Then

$$\begin{aligned}(a_1 + a_2 + a_3 + a_4)^2 &= a_1^2 + a_2^2 + a_3^2 + a_4^2 + 2 + 2a_1 a_3 + 2a_2 a_4 \\ &= (a_1 + a_3)^2 + (a_2 + a_4)^2 + 2 \\ &\geq 2[(a_1 + a_3)(a_2 + a_4)] + 2 = 4 .\end{aligned}$$

Therefore the minimum value of $a_1 + a_2 + a_3 + a_4$ is 2.

When $n \geq 5$, we have that

$$\begin{aligned}a_1 a_3 + a_2 a_4 + \cdots + a_{n-1} a_1 + a_n a_2 &\geq a_1 a_2 + a_2 a_3 + \cdots + a_{n-1}[(a_1 - a_n) + a_n] + a_n a_1 \\ &= a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n + a_n a_1 + a_{n-1}(a_1 - a_n) .\end{aligned}$$

Hence

$$\begin{aligned}(a_1+a_2+\cdots+a_n)^2 &= a_1^2+a_2^2+\cdots+a_{n-1}^2+a_n^2+2(a_1a_2+a_2a_3+\cdots+a_{n-1}a_n+a_na_1)\\ &\quad +2(a_1a_3+a_2a_4+\cdots+a_{n-1}a_1+a_na_2)+\sum\{a_ia_j : i-j\not\equiv \pm1,\pm2 \pmod n\}\\ &\geq a_1^2+a_2^2+\cdots+a_{n-1}^2+a_n^2+4(a_1a_2+a_2a_3+\cdots+a_{n-1}a_n+a_na_1)\\ &\quad +2a_{n-1}(a_1-a_n)\\ &= a_1^2+a_2^2+\cdots+a_{n-1}^2+a_n^2+4+[a_{n-1}+(a_1-a_n)]^2-a_{n-1}^2-(a_1-a_n)^2\\ &= a_2^2+\cdots+a_{n-2}^2+4+(a_{n-1}+a_1-a_n)^2+2a_1a_n \geq 4\ .\end{aligned}$$

Therefore the minimum value of $a_1+a_2+\cdots+a_n$ is equal to 2.

8. Let a, b, c be members of a real inner-product space $(V, \langle , \rangle)$ whose norm is given by $\|x\|^2 = \langle x, x\rangle$. (You may assume that V is $\mathbf{R}^n$ if you wish.) Prove that

$$\|a+b\|+\|b+c\|+\|c+a\| \leq \|a\|+\|b\|+\|c\|+\|a+b+c\|$$

for $a, b, c \in V$.

Solution. Squaring both sides of the desired inequality, we see that it suffices to prove the following

$$\|p+r\|\|q+r\| \leq \|p\|\|q\|+\|r\|\|p+q+r\| \tag{1}$$

for $p, q, r \in V$ and apply this to permutations of a, b, c.

Making the substitution $x = p+r$, $y = q+r$, $z = p+q+r$, so that $p = z-y$, $q = z-x$, $r = x+y-z$, we see that (1) is equivalent to

$$\|x\|\|y\| \leq \|z-x\|\|z-y\|+\|z\|\|x+y-z\| \tag{2}$$

for $x, y, z, \in V$.

Let w be the orthogonal projection of z onto the span of x and y. Then $z = w+v$ where v is orthogonal to x, y and w, so that

$$\|z-x\|^2 = \|(w-x)+v\|^2 = \|w-x\|^2+\|v\|^2 \geq \|w-x\|^2$$

and $\|z-x\| \geq \|w-x\|$. Similarly, $\|z-y\| \geq \|w-y\|$, $\|z\| \geq \|w\|$ and $\|x+y-z\| \geq \|x+y-w\|$. Thus, it is enough to prove that

$$\|x\|\|y\| \leq \|w-x\|\|w-y\|+\|w\|\|x+y-w\| \tag{2}$$

where x, y, w belong to a two-dimensional real inner product space.

But such a space is isometric to $\mathbf{C}$ with the usual absolute value, so we may suppose that $x, y, w \in \mathbf{C}$ and can be multiplied. Since

$$xy = (w-x)(w-y)+w(x+y-w)\ ,$$

an application of the triangle inequality yields the result.

Comment. Geometrically, equation (2) can be formulated as: *suppose that $OABC$ is a parallelogram and that P is a point in space; then $|OA||OC| \leq |PO||PB|+|PA||PC|$.*

9. Let p be a prime congruent to 1 modulo 4. For each real number x, let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Determine

$$\sum\left\{\left\{\frac{k^2}{p}\right\} : 1 \leq k \leq \frac{1}{2}(p-1)\right\}\ .$$

Solution. When x is not divisible by p, then $x \not\equiv -x \pmod{p}$, since p is odd. Therefore, the mapping $x \to x^2$ is a two-one mapping on $\mathbf{Z}_p^*$ (the integers modulo p that are coprime with p); thus, precisely $\frac{1}{2}(p-1)$ elements of $\mathbf{Z}_p^*$ are squares. We first show that m is a square modulo p if and only of $p-m$ is a square modulo p.

If s is a square in $\mathbf{Z}_p^*$ with $r^2 = s$, then $s^{-1} = (r^{-1})^2$ is also a square, so we can split the squares into disjoint sets of inverses. These disjoint sets contain precisely two elements when the square is not 1 nor -1, if applicable. Since there are an even number $\frac{1}{2}(p-1)$ of squares and 1 is a square, there must be another singleton set of inverse squares, and this can consist only of -1. Hence -1 is a square in $\mathbf{Z}_p^*$. Therefore, if m is a square in $\mathbf{Z}_p^*$, so also is $-m = (-1)m$. In other words, m is a square modulo p if and only if $p-m$ is a square modulo p.

The set $\{k^2 : 1 \le k \le \frac{1}{2}(p-1)\}$ contains each nonzero square exactly once, and so contains $\frac{1}{4}(p-1)$ pairs of the form $\{m, p-m\}$ where $1 \le m \le \frac{1}{2}(p-1)$. Since x/p is not an integer for $1 \le x \le p-1$, $\{x/p\} + \{(p-x)/p\} = 1$. Therefore

$$\sum_{k=1}^{\frac{1}{2}(p-1)} \{k^2/p\} = \sum \left\{ \{m/p\} + \{(p-m)/p\} : 1 \le m \le \frac{1}{2}(p-1), m \text{ a square mod } p \right\} = \frac{p-1}{4} .$$

10. Suppose that a path on a $m \times n$ grid consisting of the lattice points $\{(x,y)\} : 1 \le x \le m, 1 \le y \le n\}$ (x and y both integers) consisting of $mn-1$ unit segments begins at the point $(1,1)$, passes through each point of the grid exactly once, does not intersect itself and finishes at the point (m,n). Show that the path partitions the rectangle bounded by the lines $x=1$, $x=m$, $y=1$, $y=n$ into two subsets of equal area, the first consisting of regions opening to the left or up, and the second consisting of regions opening to the right or down.

Solution. Embed the grid into a rectangle composed of mn unit square cells, each with a grid point as centre. Extend the path horizontally to the left and right end of this rectangle from the respective points $(1,1)$ and (m,n). It is equivalent to show that this rectangle is decomposed by this extended path into two subsets of equal area. As the path proceeds, at any grid point, it either turns left through a right angle, or turns right through a right angle or proceeds straight ahead. Since it ends up going in the same direction as it started, it makes an equal number of left and right turns. The path keeps the region opening left or up on its left and the region opening right or down on its right. At each left turn, the path splits the area of the cell it is in into two subsets of areas $1/4$ on the left and $3/4$ on the right; at each right turn, its splits the cell in the opposite way. If the path goes straight ahead, it splits the cell area in half. It follows from this that the area on the left side of the path equals the area on the right side of the path, and the result holds.