PRODUCTS OF CONSECUTIVES THAT ARE CLOSE TO SQUARES.

A mathematical vignette

Ed Barbeau, University of Toronto

0. This investigation is designed to allow students to obtain a feel for how numbers interact together and also to illustrate the various dimensions of algebraic usage. The prerequisites are slight – just some familiarity with integer multiplication and squares and the rudiments of Grade 9 algebra. This is presented in the spirit of providing a topic that may be helpful in promoting student understanding, but in no way as a recipe for how it may be presented. The actual discussion can proceed in a number of different ways, particularly if some students asks a question, has a difficulty or makes an insightful comment. Depending on the class, it may be a matter of open-ended investigation, or the teacher may have to explicitly state a problem and then offer general guidance on how it might be pursued. There is nothing sacred about the order of presentation; some teachers may find it preferable to deal first with products of two consecutive integers.

1. A result that lends itself to investigation by secondary students is the proposition that the product of four consecutive positive integers is never square. It can be approached in a number of ways, and depending on the group, questions arise about mathematical practice that can be explored.

The result can be approached by either stating the proposition as a statement to be proved, or by asking students to write out a number of such products to see what they observe. The first few instances are

$$1 \times 2 \times 3 \times 4 = 24;$$

$$2 \times 3 \times 4 \times 5 = 120;$$

$$3 \times 4 \times 5 \times 6 = 360;$$

$$4 \times 5 \times 6 \times 7 = 840;$$

$$5 \times 6 \times 7 \times 8 = 1680.$$

Where the students go from here depends on the ability of at least some of them to recognize properties and patterns. Here the observation might be that the products are each one less than the respective squares of 5, 11, 19, 29, 41. (They might recognize only the first two and have to conjecture the others.) If we throw in $0 \times 1 \times 2 \times 3 = 0$, we get a product that is one less than the square of 1.

Recognizing that there is a pattern, we might try to see if there is a general formula for the product

$$n \times (n + 1) \times (n + 2) \times (n + 3).$$

How students arrive at a suitable formula will depend on their mathematical insight and experience. A knowledgable student will recognize that the differences for the sequence $1, 5, 11, 19, 29, 41$ form an arithmetic progression and interpolate a quadratic function. The typical high school class will not have access to this (although this situation might be a good place to introduce it).

One insightful observation is that $1 = 2^2 - 3$, $5 = 3^2 - 4$, $11 = 4^2 - 5$, $19 = 5^2 - 6$, $29 = 6^2 - 7$, $41 = 7^2 - 8$. This leads to the conjecture that

$$n \times (n+1) \times (n+2) \times (n+3) = [(n+2)^2 - (n+3)] - 1 = [n^2 + 3n + 1]^2 - 1.$$

This is readily checked. Here we see two roles of algebra, the first as a notation to describe a general pattern, and the second as a proof technique for a general result with infinitely many instances.

At this point, students who have been given the proposition will jump to the conclusion that, because the product is one less than a perfect square, it cannot itself be a square. Otherwise, the teacher may have to ask whether it is possible for the product itself to be a square.

This focusses now on the question as to whether two positive squares can differ by 1. A popular approach is for the students to observe from the sequence $\{1, 4, 9, 16, \dots\}$ that two *consecutive* squares cannot differ by 1. How can one make this precise? Again algebra can be used: $(n+1)^2 - n^2 = 2n + 1 > 1$. Fine; we have the result for consecutive positive squares. But what about *any* two positive squares? This may be splitting hairs, but it is worth having students attempt to give a succinct explicit argument.

An alternative argument starts with the equation $x^2 - y^2 = 1$ to be solved for integers x and y with $x > y$, and exploits the factorization of a difference of squares along with the solution of a simple pair of simultaneous equation, both early topics for a beginning algebra student. The equation can be rewritten

$$(x - y)(x + y) = 1,$$

the integer 1 as a product of two positive integers. The only possibility is $x + y = x - y = 1$ which forces $(x, y) = (1, 0)$. Thus, there is no solution with both x and y positive.

Returning to the fourfold product, we can draw out information by a strategic rearrangement of terms. This is a common enough process in school algebra that students should be tuned into this possibility. We can write the general product as

$$\begin{aligned} n(n+1)(n+2)(n+3) &= [n(n+3)][(n+1)(n+2)] = [n^2 + 3n][n^2 + 3n + 2] \\ &= [(n^2 + 3n + 1) - 1][(n^2 + 3n + 1) + 1] \\ &= [n^2 + 3n + 1]^2 - 1. \end{aligned}$$

One possible motivation for this bit of legerdemain might be looking at small numerical cases and noting that $1 \times 2 \times 3 \times 4 = 4 \times 6$, $2 \times 3 \times 4 \times 5 = 10 \times 12$, and generalizing.

3. Staying with the fourfold products, we see that there is an interesting connection with Pascal's triangle. For each integer greater than or equal to 4, there exists a second integer m for which

$$\binom{m}{2} = 3\binom{n}{4}.$$

This could be discovered empirically by examining the entries of Pascal's triangle, or else it could be stated and the students asked to prove it by solving an equation for m. In fact, it turns out that

$$m = \binom{n-1}{2}.$$

Now we have that

$$\begin{aligned} n(n-1)(n-2)(n-3)+1 &= 24\binom{n}{4} + 1 = 8\binom{m}{2} \\ &= 4m(m-1)+1 = (2m-1)^2 = [(n-1)(n-2)-1]^2 \\ &= (n^2-3n+1)^2. \end{aligned}$$

If we replace n by $n+3$ in this equation, then we find that

$$(n+3)(n+2)(n+1)n + 1 = (n^2+3n+1)^2,$$

as before.

3. Having disposed of this situation, students could be asked to investigate a number of questions: (a) can the product of two consecutive positive integers ever be a perfect square? (b) can the product of three consecutive positive integers ever be a perfect square? (c) is there a value of k such that some product of k consecutive positive integers is a perfect square? if so, what is the smallest such k?

Here are the points that might be drawn out for (a). Two consecutive positive integers are coprime (*i.e.* have greatest common divisor 1), so that if their product is square, then each of them must be square. One way of getting at this is to consider the prime factor decomposition and note that squares are characterized by the fact that all their prime divisors do so to an even exponent.

Alternatively, we can note that $n(n+1)$ lies between two consecutive squares, and so cannot be square. This is straightforward, since

$$n^2 < n(n+1) = n^2 + n < (n+1)^2.$$

Another approach takes us into territory we have already visited. Any number x is a square if and only if $4x$ is also a square. Since

$$4n(n+1) = (2n+1)^2 - 1,$$

and two positive squares cannot differ by 1, then $4n(n+1)$ and with it $n(n+1)$ cannot be square.

Because $n(n+1)(n+2)$ is a polynomial of odd degree in n, there is no obvious way to approach the possibility of its being square using elementary algebra. However, we can observe that any odd prime can divide at most one of the three factors, and so must do so to an even degree. Furthermore, when n is odd, the prime 2 can divide only the middle factor, so that the three factors are pairwise coprime. Thus, if the triple product is square in this case, each of its three factors must also be square, an impossibility.

However, if n is even, then 2 divides both n and $n+2$ and divides exactly one of these to the first power. The bottom line here is that both $n+1$ and $n(n+2)$ are squares. Using the fact that $n(n+2) = (n+1)^2 - 1$, we again find that the triple product cannot be a positive square.

As for (c), this is a deeper problem to be explored by students, to see what properties a square which is the product of k consecutive positive integers must have.

4. In this section, I will some numerical results that might be useful in the investigation. To begin with, the question as to whether any positive square can be represented by a product of two or more consecutive integers has been settled in the negative by Paul Erdös in the paper:

Paul Erdös, *Notes on products of consecutive integers.* Jour. London Math, Soc. 14 (1939), 194-198

For each positive integer k with $k \geq 2$, let $g_k(n)$ be the product of k consecutive integers, the smallest of which is n and let $f_k(n)^2$ be the smallest square that exceeds $g_k(n)$.

Product of three consecutive integers

$g_3(n) = n(n+1)(n+2) = n^3 + 3n^2 + 2n$. When $n = m^2 - 1$, $g_3(n) = m^6 - m^2$, and

$$(m^3 - 1)^2 < m^6 - m^2 < m^6,$$

so $f_3(m^3 - 1) = m^3$ and $f_3(m^2 - 1)^2 - g_3(m) = m^2$. Thus when $n = m^2 - 1$, n differs from the next greater square by a square. This covers $n = 3, 8, 15, 24, \ldots$.

More generally, we have

n	$g_3(n)$	$f_k(n)$	$f_k(n)^2 - g^n(k)$	Later differences
$m^2 - 1$	$m^6 - m^2$	m^3	m^2	$2m^3 + m^2 + 1$, $4m^3 + m^2 + 4$
$4m^2 - 2$	$64m^6 - 48m^4 + 8m^2$	$8m^3 - 3m$	m^2	
$m^2 - 3$	$m^6 - 6m^4 + 11m^2 - 6$	$m^3 - 3m$	$2m^2 - 6$	
$m^2 + 1$	$m^2 + 6m^4 + 11m^2 + 6$	$m^3 + 3m$	$2m^2 + 6$	
$4m^2$	$64m^2 + 48m^4 + 8m^2$	$8m^3 + 3m$	m^2	

This covers the cases $n = 2, 3, 4, 5, 6, 8, 10, 13, 14, 15, 16, 17, 22, 24, 26, 33, 34, 35, 36, 37, 46, 48, 50, 61, 62, 63, 64, 6$

n	$g_3(n)$	$f_k(n)$	$f_k(n)^2 - g^n(k)$	Later differences
1	6	3	3	13, 22, 33
2	24	5	$\mathbf{1 = 1^2}$	12, $\mathbf{25 = 5^2}$
3	60	8	$\mathbf{4 = 2^2}$	21, 40
4	120	11	$\mathbf{1 = 1^2}$	24, $\mathbf{49 = 7^2}$, 76
5	210	15	15	46, 79
6	336	19	$\mathbf{25 = 5^2}$	$\mathbf{64 = 8^2}$, 105
7	504	23	$\mathbf{25 = 5^2}$	
8	720	27	$\mathbf{9 = 3^2}$	$\mathbf{64 = 8^2}$, $\mathbf{121 = 11^2}$
9	990	32	34	
10	1320	37	$\mathbf{49 = 7^2}$	
11	1716	42	48	
12	2184	47	$\mathbf{25 = 5^2}$	
13	2730	53	79	
14	3360	58	$\mathbf{4 = 2^2}$	$\mathbf{121 = 11^2}$, 240, $\mathbf{361 = 19^2}$
15	4080	64	$\mathbf{16 = 4^2}$	
16	4896	70	$\mathbf{4 = 2^2}$	
17	5814	77		
18	6480	83	$\mathbf{49 = 7^2}$	
19	7980	90		
20	9240	97	$\mathbf{169 = 13^2}$	
21	10626	104	190	
22	12144	111	177	
23	13800	118		$\mathbf{361 = 19^2}$
24	15600	125	$\mathbf{25 = 5^2}$	
25	17550	133	139	
26	19656	141	$\mathbf{225 = 15^2}$	
27	21924	149		$\mathbf{576 = 24^2}$
28	24360	157	$\mathbf{289 = 17^2}$	
29	26970	165	745	
30	29760	173	$\mathbf{169 = 13^2}$	
31	32736	181	$\mathbf{25 = 5^2}$	
32	35904	190	$\mathbf{196 = 14^2}$	577, 960
33	39270	199	331	
34	42840	207	$\mathbf{9 = 3^2}$	− , $\mathbf{841 = 29^2}$
35	46620	216	$\mathbf{36 = 6^2}$	
36	50616	225	$\mathbf{9 = 3^2}$	
37	54834	235	391	
38	59280	244	$\mathbf{256 = 16^2}$	
39	63960	253	$\mathbf{49 = 7^2}$	
40	68880	263	$\mathbf{289 = 17^2}$	
41	74046	273	483	
42	79464	282	60	$\mathbf{625 = 25^2}$

Product of four consecutive integers

We have

$$\begin{aligned} g_4(n) &= (n^2+3n+1)^2 - 1 \\ &= (n^2+3n+2)^2 - 2(n^2+3n+2) = (n^2+3n+2)^2 - 2(n+1)(n+2) \\ &= (n^2+3n+3)^2 - (4n^2+12n+9) = (n^2+3n+3)^2 - (2n+3)^2 \end{aligned}$$

Product of five consecutive integers

n	$g_5(n)$	$f_5(n)$	$(f_5(n))^2 - g_5(n)$	Subsequent differences
1	120	11	$\mathbf{1^2}$	24, $\mathbf{7^2}$, 76, 105
2	720	27	$\mathbf{3^2}$	$\mathbf{8^2}$, $\mathbf{11^2}$, 180, 241
3	2520	51	$\mathbf{9^2}$	184, $\mathbf{17^2}$, 396
4	6720	82	$\mathbf{2^2}$	$\mathbf{13^2}$, 336, 505
5	15120	123	$\mathbf{3^2}$	$\mathbf{16^2}$, 505, 756, 1009
6	30240	174	$\mathbf{6^2}$	385, 736, $\mathbf{33^2}$, $\mathbf{38^2}$
7	55440	236	$\mathbf{16^2}$	$\mathbf{27^2}$, 1204, $\mathbf{41^2}$, 2160
8	95040	309	$\mathbf{21^2}$	1060, $\mathbf{41^2}$, 2304, 2929
9	154440	393	$\mathbf{3^2}$	796, 1585, 2376
10	240240	491	$\mathbf{29^2}$	1824, $\mathbf{53^2}$, 3796
11	360360	601	$\mathbf{29^2}$	2044, $\mathbf{57^2}$, 4456, 5665
12	524160	724	$\mathbf{4^2}$	1465, 2916, 4369
13	724560	862	$\mathbf{22^2}$	$\mathbf{47^2}$, 3936, 5665
14	1028160	1014	$\mathbf{6^2}$	2065, $\mathbf{64^2}$, 6129
15	1395360	1182	$\mathbf{42^2}$	4129, 6496, 8865
16	1860480	1364	$\mathbf{4^2}$	2745, $\mathbf{74^2}$, 18209
17	2441880	1563	$\mathbf{33^2}$	4216, 7345, 10476
18	3160080	1778	1204	$\mathbf{69^2}$, 8320, $\mathbf{109^2}$
19	4037880	2010	2220	$\mathbf{79^2}$, 34444, 38481
20	5100480	2259	$\mathbf{51^2}$	7120, 11641, 16164
21	6375600	2525	$\mathbf{5^2}$	5076, 10129, 15184
22	7893600	2810	$\mathbf{50^2}$	8121, 13744, 19369
23	9687600	3113	3169	9396, $\mathbf{125^2}$, 21956
24	11793600	3435	$\mathbf{75^2}$	12496, 19369, $\mathbf{162^2}$
25	14250600	3775	$\mathbf{5^2}$	7576, $\mathbf{123^2}$, 22684
26	17100720	4136	$\mathbf{76^2}$	14049, 22324, 30601

Product of six consecutive integers

We have

$$\begin{aligned} g_6(n) &= (n^3+8n^2+15n)(n^3+7n+14n+8) \\ &\quad = [\tfrac{1}{2}(2n^3+15n^2+29n+8)]^2 - [\tfrac{1}{2}(n^2-n-8)]^2 \\ &= (n^3+8n^2+17n+10)(n^3+7n+12n) \\ &\quad = [\tfrac{1}{2}(2n^3+15n^2+29n+10)]^2 - [\tfrac{1}{2}(n^2+5n+10)]^2 \\ &= (n^3+8n^3+19n+12)(n^3+7n^2+10n) \\ &\quad = [\tfrac{1}{2}(2n^3+15n^2+29n+12)]^2 - [\tfrac{1}{2}(n^2+9n+12)]^2 \end{aligned}$$

We note that the polynomials $f_6(n)$, $f_6(n)+1$, $f_6(n)+2$ yielding squares exceeding $g_6(n)$ and the corresponding square roots of the minuends have non-integer coefficients. However, since they affect the coefficients of n^2 and n which have the same parity, the polynomials always take integer values.

Products of seven consecutive integers

n	$g_7(n)$	$f_7(n)$	$(f_7(n))^2 - g_7(n)$	Subsequent differences
1	5041	71	$\mathbf{1^2}$	143, 288, 435, 584
2	40320	201	$\mathbf{9^2}$	$\mathbf{22^2}$, 889, $\mathbf{36^2}$, 1705
3	181440	426	$\mathbf{6^2}$	889, 1744
4	604800	778	$\mathbf{22^2}$	2041
5	1663200	1290	$\mathbf{30^2}$	$\mathbf{59^2}$, 6064
6	3991680	1998	$\mathbf{18^2}$	4321
7	8648640	2941	$\mathbf{29^2}$	$\mathbf{82^2}$
8	17297280	4159	$\mathbf{1^2}$	8320, $\mathbf{129^2}$
9	32432400	5695	$\mathbf{25^2}$	12016
10	57657600	7594	$\mathbf{106^2}$	26425
11	98017920	9901	$\mathbf{109^2}$	
12	160392960	12665	9265	$\mathbf{186^2}$
13	253955520	15936	$\mathbf{24^2}$	
14	390700800	19767	$\mathbf{183^2}$	
15	586051200	24209	24481	$\mathbf{270^2}$, 121321
16	859541760	29318	$\mathbf{58^2}$	
17	1235591280	35151	$\mathbf{39^2}$	
18	1744364160	41766	$\mathbf{186^2}$	
19	2422728000	49222	$\mathbf{278^2}$	
20	3315312000	57579	$\mathbf{171^2}$	
21	4475671200	66901	72601	206404, 340209

Product of eight consecutive integers

We have

$$\begin{aligned} g_8(n) &= (n^4+14n^3+63n^2+98n+28)^2 - 16(2n+7)^2 \\ &= (n^4+14n^3+63n^2+98n+30)^2 - 4(2n^2+7n+15)^2 \\ &= (n^4+14n^3+63n^2+98n+36)^2 - 16(n^2+7n+9)^2 \end{aligned}$$

The smallest square bigger than $g_8(n)$ is not equal to $(n^2{+}14n^3{+}63n^2{+}98n{+}28)^2$ until $n \geq 4$.

$$g_8(1) = 40320 = 201^2 - 9^2 = 202^2 - 22^2 = 203^2 - 889 = 204^2 - 36^2.$$

$$g_8(2) = 362880 = 603^2 - 27^2 = 604^2 - 44^2 + 605^2 - 3145 = 606^2 - 66^2.$$

$$g_8(3) = 1814400 = 1347^2 - 3^2 = 1348^2 - 52^2 = 1349^2 - 5401 = 1350^2 - 90^2.$$

$$g_8(4) - 6652800 = 2580^2 - 60^2 = 2582^2 - 118^2.$$

Product of nine consecutive integers

n	$g_9(n)$	$f_9(n)$	$(f_9(n))^2 - g_7(n)$	Subsequent differences
1	362880	603	$\mathbf{27^2}$	$\mathbf{44^2}$, 3145, $\mathbf{66^2}$, 5569
2	3628800	1905	$\mathbf{15^2}$	4036, 7879, $\mathbf{108^2}$, 15481
3	19958400	4468	$\mathbf{68^2}$	13561, $\mathbf{150^2}$, 31441, 40384
4	79833600	8935	$\mathbf{25^2}$	$\mathbf{136^2}$, 36369, 54244
5	259459200	16108	$\mathbf{92^2}$	
6	726485760	26954	32356	86265
7	1816214400	42618	$\mathbf{282^2}$	
8	4151347200	64431	$\mathbf{81^2}$	
9	8821612800	93924	$\mathbf{324^2}$	
10	$1.76432256 \times 10^{10}$	132828	$\mathbf{228^2}$?	