SQUARE-PAIR NUMBERS.

A mathematical vignette

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The positive integer n is said to be an SP (square-pair) number if the set $\{1, 2, \ldots, 2n\}$ can be partitioned into n pairs such that the sum of the numbers in each pair is a perfect square.

For example, 8 is a SP number with the partition

$$(16, 9), (15, 10), (14, 11), (13, 12), (8, 1), (7, 2), (6, 3), (5, 4).$$

However, 10 is not a SP number. Here is why.

Suppose, if possible, there is a partition of $\{1, 2, \ldots, 20\}$ that fulfils the condition for 10 to be an SP number. The partition must include the pair $(18, 7)$. The only possible pairs that include 4 are $(4, 5)$ and $(4, 12)$. If the set of pairs include $(18, 7)$ and $(4, 5)$, then $(9, 7)$ and $(20, 5)$ are ruled out, and so that leaves $(9, 16)$ and $(20, 16)$. Since both of these pairs cannot be included, we are at an impasse.

On the other hand, if $(18, 7)$ and $(4, 12)$ are included, then $(9, 7)$ and $(13, 12)$ are excluded, and we must include $(9, 16)$ and $(13, 3)$. But then $(20, 16)$ is excluded and we must include $(20, 5)$. Since $(11, 5)$ is now excluded, we must include $(11, 14)$. But then $(2, 7)$ and $(2, 14)$ are excluded, and we cannot have a pair including 2.

Here are a few questions to be investigated:

- Which numbers are SP and which are not? Are there infinitely many numbers in each category?
- Characterize the numbers for which all pairs in the partition have the same sum.
- Can you find other families of SP numbers for which the square sums of the pairs in the partition are different.

SP numbers with equal pair sums

First, we look at SP numbers for which all the pairs in the partition add to the same square, s^2. Then

$$ns^2 = 1 + 2 + \cdots + 2n = n(2n + 1)$$

from which $2n+1 = s^2$. Thus s^2 is odd, and has the form $(2m+1)^2 = 4m^2+4m+1$. Hence $n = 2m(m + 1)$.

On the other hand, if $n = 2m(m + 1)$, then the partition

$$((2m + 1)^2 - 1, 1), ((2m + 1)^2 - 2, 2), \ldots (2m^2 + 2m + 1, 2m^2 + 2m)$$

is a partition of the desired type.

There is another way to identify the form of n. Suppose that n is a SP number and we have the pairs $(1, a)$ and $(b, 2n)$, which could be the same. If $1+a = b+2n$, then $a - b = 2n - 1$. Since $b \leq 2n$ and $b \geq 1$, then $a - b \leq 2n - 1$ with equality if and only if $a = 2n$ and $b = 1$. Therefore $1 + 2n$ is an odd square $(2m + 1)^2 = 4m(m + 1) + 1$, from which we find that $n = 2m(m + 1)$. We can finish as in the last paragraph.

Families of SP numbers

1. Suppose that we have found a suitable partition for the set $\{1, 2, \dots, 2m\}$. Then for each sufficiently large $2k(k + 1) - m$ is a SP number. For the numbers above $2m$, the pairs start with $(2m + 1, 4k^2 + 4k - 2m)$. The SP numbers generate others:

$$0 : 4, 12, 24, \dots$$
$$4 : 8, 20, 36, 56, \dots$$
$$7 : 17, 33, 53, 77, \dots$$
$$8 : 16, 32, 52, \dots$$
$$9 : 15, 31, \dots$$

2. [P. Chaitin] Let $n = 2m(3m + 1)$ so that $2n = 4m(3m + 1) = 12m^2 + 4m = (4m + 1)^2 - (2m + 1)^2$. Then n is a SP number with pairs $(k, (2m + 1)^2 - k)$ with $1 \leq k \leq 2m(m+1)$ and $((2m+1)^2+k, (4m+1)^2-(2m+1)^2-k)$ for $0 \leq k \leq 4m^2-1$.

3. Begin with the square m^2 and, where possible, select a square r^2 that satisfies the two conditions (1) $m^2 \leq 2(r^2 - 1)$ and (2) $3r^2 \leq 2m^2$. Then we have the following partition for $n = m^2 - r^2$.

$$(r^2 - 1, 1), (r^2 - 2, 2), \dots, (m^2 - r^2 + 1, 2r^2 - m^2 - 1),$$
$$(2(m^2 - r^2), 2r^2 - m^2), (2(m^2 - r^2) - 1, 2r^2 - m^2 + 1), \dots, (r^2, m^2 - r^2).$$

The conditions on m and r ensure that for each of the two chunks of pairs, the first entries decrease and the second entries increase.

It remains to ensure that for sufficiently large m, we can find r such that

$$\frac{1}{2}(m^2 + 1) \leq r^2 \leq \frac{2}{3}m^2.$$

But this follows from the fact that the difference between the square roots of the extremes of the inequality is greater than 1 for sufficiently large m and so contains an integer.

4. Inspired by the partition

$$(14, 2), (13, 3), (12, 4), (11, 5), (10, 6), (9, 7), (8, 1)$$

for $n = 7$, we let n be of the form $2m^2 - 1$ and consider the partition consisting of the $2m^2 - 2$ pairs $(4m^2 - k, k)$ for $2 \leq k \leq 2m^2 - 1$, and $(2m^2, 1)$ This will be an acceptable partition provided $2m^2 + 1 = r^2$ for some integer r. But this is a Pell's equation and so has infinitely many solutions.

For small values of n, it is straightforward to check that 1, 2, 3, 5, 6 are SP while 4, 7, 8, 9 are. Sometimes it is not so easy to fit an example into a larger pattern. In the case of 9, we have the partition:

$$(18,7),(17,8),(16,9),(15,1),\ldots,(10,6).$$

ST (square triple) numbers

We can also define ST numbers n for which the set $\{1,2,\ldots,3n\}$ can be partitioned into triplets for which the sum of the numbers in each triplets is a square. So far there are two ST numbers, 5 and 7.

$$5:(15,13,8),(14,12,10),(11,4,1),(9,5,2),(7,6,3)$$

$$7:(21,17,11),(20,16,13),(19,18,12),(15,9,1),(14,6,5),(10,8,7),(4,3,2)$$