Triangles with a 60° or 120° angle.

A mathematical vignette

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1. The Diophantine equations.

The By the Law of Cosines, a triangle with sides a, b, c and an angle of 120° opposite side c has its sides related by the equation

$$a^2 + ab + b^2 = c^2.$$

If the angle opposite side c is 60°, then the sides satisfy the equation

$$a^2 - ab + b^2 = c^2.$$

Let $f(a, b) = a^2 + ab + b^2$ and $g(a, b) = a^2 - ab + b^2$. Then it can be verified that

$$f(a, b) = g(a, a + b) = g(b, a + b).$$

Analogously to Pythagorean triples, we can look for triples (a, b, c) of integers that are the sides of triangles containing an angle of either 120° or 60°. Thus, we want to find integer solutions for the equations

$$f(a, b) = c^2 \qquad \text{and} \qquad g(a, b) = c^2.$$

These equations can be written, respectively, as

$$(2a + b)^2 + 3b^2 = (2c)^2$$

and

$$(2a - b)^2 + 3b^2 = (2c)^2.$$

Thus, we can obtain solutions to the foregoing equation by solving the Diophantine equation

$$u^2 + 3v^2 = w^2$$

where w is even, and setting $c = w/2$, $a = v$ and $b = (u \pm v)/2$.

For example, $(a, b, c) = (3, 5, 7)$ satisfies

$$a^2 + ab + b^2 = c^2$$

while $(a, b, c) = (3, 8, 7)$ and $(a, b, c) = (5, 8, 7)$ satisfy

$$a^2 - ab + b^2 = c^2.$$

The corresponding solutions to $u^2 + 3v^2 = w^2$ are

$$(u, v, w) = (13, 3, 14), (13, 3, 14), (11, 5, 14).$$

An alternative approach is to imitate the process for getting a general solution in integers to the Pythagorean equation. We look for primive solutions of $a^2+ab+b^2 = c^2$ in which the greatest common divisor of a, b, c is 1. We note that, in this case, c cannot be a multiple of 3. Suppose otherwise. Then $(2c)^2$ is divisible by 9, as is $(2a + b)^2 + 3b^2$. But then $2a + b$ is a multiple of 3, and so is b. Hence a and b are both divisible by 3.

We note that, for a primitive solution, at least one of a and b must be odd; suppose that it is b. Rewriting the equation as

$$3b^2 = (2c)^2 - (2a+b)^2 = [2c - (2a+b)][2c + (2a+b)],$$

we note that both factors on the right must be odd, and that the square of any odd common divisor d of them must divide $3b^2$.

Modulo 3, $(2c)^2 \equiv (2a+b)^2 \equiv 1$, and $2a+b \equiv -(2b+1)$. Hence, either $2c-(2a+b)$ or $2c-(2b+a)$ is divisible by 3.

Since d divides both $2c-(2a+b)$ and $2c+(2a+b)$, then d must divide b as well their sum $4c$. But then d divides b and c and so divides a; thus $d = 1$. Therefore, either

$$2c - (2a+b) = 3y^2 \qquad \text{and} \qquad 2c + (2a+b) = x^2$$

for some odd integers x and y,or

$$2c + (2a+b) = x^2 \qquad \text{and} \qquad 2c - (2a+b) = 3y^2$$

for some odd integers x and y.

Solving the first system for a, b, c yields $4c = x^2 + 3y^2$, $b = xy$ and $4a = x^2 - 2xy - 3y^2 = (x-y)^2 - 4y^2$. For $a \geq 0$, we require that $x \geq 3y$.

We can check that

$$2c - (2a+b) = \frac{1}{4}(8c - 8b - 4a) = \frac{1}{4}(2x^2 + 6y^2 - 2x^2 + 4xy + 6y^2 - 4xy) = 3y^2,$$

and, which we will need later,

$$2c + a - b = 3\left[\frac{(x-y)}{2}\right]^2.$$

Alternatively, solving the second system for a, b, c yields $4c = x^2 + 3y^2$, $b = xy$ and $4a = 3y^2 - x^2 = 4y^2 - (x-y)^2$. In this case,

$$2c - (2b+a) = \frac{1}{4}(8c - 8b - 4a) = \frac{1}{4}(3x^2 - 6xy + 3y^2) = 3\left[\frac{x-y}{2}\right]^2$$

and

$$2c + b - a = \frac{1}{4}(8c + 4b - 4a) = \frac{1}{4}(3x^2 + 6xy + 3y^2) = 3\left[\frac{x+y}{2}\right]^2.$$

The consequence of this is that, for any triple representing a 120° triangle, we can order the shorter sides so that $2c - (2a+b) = 3y^2$, $2c + (2a+b) = x^2$ and $2c + a = 3[(x-y)/2]^2$ for some x and y. The quantities x and y will have the same parity as b. For example, when $(a,b,c) = (8,7,13)$, then $2c - (2a+b) = 3 \times 1^2$ and $2c + a - b = 3 \times 3^2$, while when $(a,b,c) = (5,16,19)$, $2c - (2a+b)3 \times 2^2$ and $2c + a - b = 3 \times 3^2$.

2. Another approach to $a^2 + ab + b^2 = c^2$.

A systematic way of generating solution to $c^2 = a^2 + ab + b^2$ arises from the observation that the right side is equal to $(a+b)^2 - ab$ and so less that $(a+b)^2$. Write

$$a^2 + ab + b^2 = (a + b - k)^2$$

for some positive value of k.

The equation

$$a^2 + ab + b^2 = (a + b)^2 - ab = (a + b - k)^2$$

can be simplified to

$$(a - 2k)(b - 2k) = 3k^2.$$

There are three obvious factorizations of the right side that we can use on the left to get possible values of a and b:

$$(a - 2k, b - 2k) = (1, 3k^2),$$

$$(a - 2k, b - 2k) = (3, k^2),$$

$$(a - 2k, b - 2k) = (k, 3k).$$

These lead to the solutions

$$(a, b, c) = (2k+1, 3k^2+2k, 3k^2+3k+1) = ((k+1)^2-k^2, (2k+1)^2-(k+1)^2, (k+1)^3-k^3),$$

$$(a, b, c) = (3k, 5k, 7k),$$

and

$$(a, b, c) = (2k + 3, k^2 + 2k, k^2 + 3k + 3).$$

Replacing k by $k - 1$ in the last leads to

$$(a, b, c) = (2k + 1, k^2 - 1, k^2 + k + 1).$$

For some values of k, there will be other factorizations of $3k^2$ that will lead to other solutions.

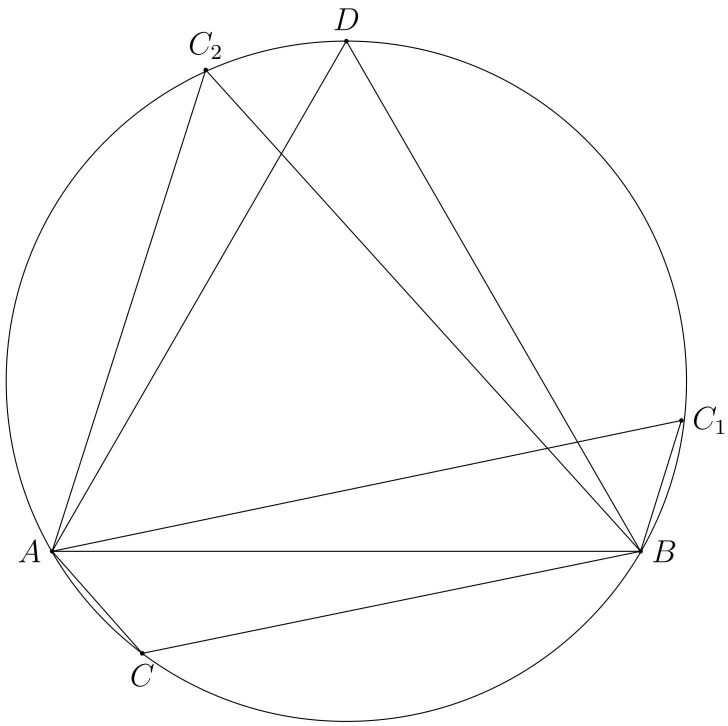
Here are some numerical solutions:

$$\begin{aligned}(a, b, c) =&(3, 5, 7), (5, 16, 19), (7, 8, 13), (7, 33, 37),\\ &(9, 56, 61), (11, 24, 31), (11, 85, 91), (16, 39, 49).\end{aligned}$$

3. Geometry.

The following diagram illustrates how the 120° and 60° triangles are related. Inscribed in the circle is an equilateral triangle whose side length is c.

The triangles



Triangle DBC is equilateral, and the length of AB is C. The $120°$ triangle is ABC and the two corresponding $60°$ triangles are ABC_1 and ABC_2. The segment AC_1 is parallel to CB, and BC_2 is parallel to CA. Observe that the lengths of both AC_1 and BC_2 is the sum of the lengths of AC and CB.

4. Cube roots of unity and a law of composition.

Let ω be a nonreal cube roots of unity, which satisfies the equations $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$. Observe that

$$f(a,b) = a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2).$$

We begin by observing that

$$\begin{aligned}(a_1 + b_1\omega)(a_2 + b_2\omega) &= a_1a_2 + b_1b_2\omega^2 - (a_1b_2 + a_2b_1)\omega \\ &= a_1a_2 - b_1b_2(1+\omega) - (a_1b_2 + a_2b_1)\omega \\ &= (a_1a_2 - b_1b_2) - (a_1b_2 + a_2b_1 + b_1b_2)\omega.\end{aligned}$$

with a similar equation with ω replaced by ω^2. From these equations, we deduce that

$$f(a_1,b_1)f(a_2,b_2) = f(a_1a_2 - b_1b_2, a_1b_2 + a_2b_1 + b_1b_2),$$

a fact that can be verified directly.

For two solutions (a_1, b_1, c_1) and (a_2, b_2, c_2) of the equation

$$a^2 + ab + b^2 = c^2$$

, we define the operation $*$ by

$$(a_1, b_1, c_1) * (a_2, b_2, c_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1 + b_1b_2, c_1c_2)$$

which yields a third solution on the right side. When $a_1a_2 > b_1b_2$ (which can be arranged by reordering the values of a_i and b_i if necessary), we can obtain from two triangles with a 120° angle and integer sides, a third such triangle. (In the case that $a_1a_2 < b_1b_2$, we can obtain a value of $g(a, b)$ in positive integers and thus a 60° triangle.)

However, this operation also provides us with a tool to generate freely infinitely many such triangles from values of $f(r, s)$, for arbitrary integers r and s, even if it is nonsquare. For, from the converse of the cosine law, any triangle with sides $(a, b, \sqrt{a^2 + ab + b^2})$ with $a > b$ has a 120° angle, as does the triangle with sides

$$(r, s, \sqrt{r^2 + rs + s^2}) * (r, s, \sqrt{r^2 + rs + s^2}) = (r^2 - s^2, 2rs + s^2, r^2 + rs + s^2).$$

Taking $r = 2$ and $s = 1$, for example, yields the triangle with sides $(3, 5, 7)$.

Does every 120° triangle arise in this way? If (a, b, c) are the sides of the triangle, then we have to solve the system

$$r^2 - s^2 = a; \qquad 2rs + s^2 = b$$

for positive integers r and s. Since $r = \sqrt{a + s^2}$, we have that $2s\sqrt{a + s^2} = b - s^2$. Squaring and rearranging terms leads to

$$3s^4 + (4a + 2b)s^2 - b^2 = 0.$$

Thus s^2 is the positive root of

$$3t^2 + 2(2r + s)t - b^2 = 0.$$

The discriminant of this quadratic is

$$4[(2a + b)^2 + 3b^2] = 16(a^2 + ab + b^2) = 16c^2$$

and its positive root is

$$\frac{1}{3}[2c - (2a + b)].$$

Thus, the system is solvable for integers r and s when

$$s^2 = \frac{1}{3}[2c - (2a + b)]$$

and

$$r^2 = a + s^2 = \frac{1}{3}[2c + a - b]$$

are both squares. We have seen that we can order the sides a and b so that $2c - (2a + b) = 3y^2$. In this case

$$4(a, b, c) = (x^2 - 2xy - 3y^2, 4xy, 4x^2 + 12y^2),$$

so that and $\frac{1}{3}(2c + a - b)$ is square. Thus, we can get all the 120° triangles by this "squaring" operation.

We can use this "squaring" approach to get parameterized families. If $(a, b) = (t, 1)$, we get the family of triangles

$$(t^2 - 1, 2t + 1, t^2 + t + 1)$$

whose first few members are $(3, 5, 7)$, $(8, 7, 13)$, $(15, 9, 21)$, $(24, 11, 31)$. If $(a, b) = (t + 1, t)$, we get the family

$$(2t + 1, t(3t + 2), 3t(t + 1) + 1)$$

whose first few members are $(3, 5, 7)$, $(5, 16, 19)$, $(7, 33, 37)$, $(11, 85, 91)$.

Correspondingly, there are parameterized families of 60° triangles with integer sides:

$$(2t + 1, (t + 1)^2 - 1, t^2 + t + 1),$$
$$(t^2 - 1, (t + 1)^2 - 1, t^2 + t + 1),$$
$$(2t + 1, (t + 1)(3t + 1), 3t(t + 1) + 1),$$
$$(t(3t + 2), (t + 1)(3t + 1), 3t(t + 1) + 1.$$

The function $f(a, b)$ has another interesting property that allow us to obtain parameterized families of triangle. Since

$$f(t - 1, 1)f(t, 1) = (t^2 - t + 1)(t^2 + t + 1) = t^4 + t^2 + 1 = f(t^2, 1),$$

we have the 120° triangle

$$(t - 1, 1, \sqrt{t^2 - t + 1}) * (t, 1, \sqrt{t^2 + t + 1}) * (t^2, 1, \sqrt{t^4 + t^2 + 1})$$
$$= (t^2 - t - 1, 2t, \sqrt{t^4 + t^2 + 1}) * (t^2, 1, \sqrt{t^4 + t^2 + 1})$$
$$= (t^4 - t^3 - t^2 - 2t, 2t^3 + t^2 + t - 1, t^4 + t^2 + 1) = ((t(t-2)(t^2+t+1), (2t-1)(t^2+t+1), (t^2-t+1)(t^2+t+1)$$

These are similar to $(t(t - 2), 2t - 1, t^2 - t + 1)$, which we essentially have found in another way.

5. 120° triangles with consecutive integer sides.

Suppose that $b = a+1$. After multiplying by 4, the equation $f(a,b) = c^2$ becomes

$$3(2a+1)^2 + 1 = (2c)^2.$$

Let $x = 2c$ and $y = 2a+1$. Then we are looking for solutions of the pellian equation $x^2 - 3y^2 = 1$. The fundamental solution of this equation is $(x,y) = (2,1)$ and the general solution in positive integers is given by $(x,y) = (x_n, y_n)$ $(n \geq 0)$, where

$$x_n + y_n\sqrt{3} = (2+\sqrt{3})^n.$$

Thus, we have the sequence of solutions

$$(x,y) = (1,0), (2,1), (7,4), (26,15), (97,56), (362,209), (1351,780), (5042,2911), \ldots.$$

The sequence $\{x_n\}$ and $\{y_n\}$ satisfy the recursions

$$x_{n+1} = 4x_n - x_{n-1}, \qquad y_{n+1} = 4y_n - y_{n-1},$$

$$x_{n+1} = 2x_n + 3y_n, \qquad y_{n+1} = x_n + 2y_n,$$

for all positive integers n. Since $x = 2c$, we are interested in only those solutions with positive values of x. The first two 120° triangles obtained in this way are $(7,8,13)$ and $(104,105,181)$, with the corresponding 60° triangles $(7,15,13)$, $(8,15,13)$, $(104,209,181)$, $(105,209,181)$.

6. Solutions of $u^2 + 3v^2 = w^2$.

In section 1, we have seen that the triangles with integer sides are related to solutions of $u^2 + 3v^2 = w^2$, where w is even. (If we have a solution with odd w, we can get one with even w by multiplying each variable by 2. Note that, if w is even, then u and v must have the same parity.

By starting with a solution of this Diophantine equation, finding the related 120° triangle and then considering its analogous 60° triangle, we are led to define the following operation which takes solution to other solutions:

$$U(u,v,w) = \left(\frac{u+3v}{2}, \frac{|u-v|}{2}, w\right),$$

$$V(u,v,w) = \left(\frac{|3v-u|}{2}, \frac{u+v}{2}, w\right).$$

This takes integer solutions with w even to other integer solutions.

When $u \geq v$, then $U^2(u,v,w) = (u,v,w)$, and when $v \geq u$, then $V^2(u,v,w) = (u,v,w)$.

We can also define a second operation on the triples of solutions to $u^2 + 3v^2 = w^2$. We can rewrite the equation as a Pell's equation $w^2 - 3v^2 = u^2$ with fixed u, and note that if the equation is satisfied by (u,v,w), it is also satisfied by $(u, 2v+w, 2w+3v)$. This allows us to construct infinitely many triangles.

For example: $W(11,5,14) = (11,24,43)$. To construct a triangle, we consider instead $(22,48,86)$ to yield the 60° triangles $(13,48,43)$ and $(35,48,43)$ and the 120° triangle $(13,35,43)$.

7. table of triangles.

Side opposite angle	120° triangles	60° triangles
7	(3, 5, 7)	(3, 8, 7), (5, 8, 7)
13	(7, 8, 13)	(7, 15, 13), (8, 15, 13)
19	(5, 16, 19)	(5, 21, 19), (16, 21, 19)
31	(11, 24, 31)	(11, 35, 31), (24, 35, 31)
37	(7, 33, 37)	(7, 40, 37), (33, 40, 37)
43	(13, 35, 43)	(13, 48, 43), (35, 48, 43)
49	(16, 39, 49)	(16, 55, 49), (39, 55, 49)
181	(104, 105, 181)	(104, 209, 181), (105, 209, 181)

Solutions of $u^2 + 3v^2 = w^2$	Solutions of $u^2 + 3v^2 = w^2$
(1, 1, 2)	
(11, 5, 14), (13, 3, 14)	(1, 4, 7)
(1, 15, 26), (23, 7, 26)	(11, 4, 13)
(37, 5, 38)	(13, 8, 19)
	(11, 24, 43)
	(47, 8, 49)
(73, 7, 74)	

8. Questions.

1. Do we get all the triangles from the solutions of $u^2 + 3v^2 = w^2$.

2. Is the side opposite the angle of a primitive triangle always 1 more than a multiple of 6? What are the possible values of the longest sides?

3. Can we "generate" all the triangles from a single one in some way?