Two periodic sequences.

A mathematical vignette

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0. Introduction.

This is an investigation that would be suitable for middle school students. It involves some facility with arithmetic, particularly in dealing with fractions. Students get some experience in recognizing patterns and expressing them mathematically. Finally, there is sharpening the ability to perform accurate algebraic manipulations.

The project involves three related sequences $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$.

1. The sequence $\mathfrak{A}$.

Suppose that you start with any two numbers a and b. These are the first two entries of a sequence $a, b, c, d, e, \ldots$ for which each successive term is the previous term plus 1 divided by the term before than. Thus

$$c = \frac{b+1}{a}, \qquad d = \frac{c+1}{b}, \qquad e = \frac{d+1}{c}$$

and so on. Start with any two numbers and see what the generated sequence looks like.

More formally, the (bilateral) sequence can be defined by

$$x_{n+1} = \frac{x_n + 1}{x_{n-1}},$$

where we allow n to range over all integers.

If a and b are positive, then all subsequent terms will be positive and so the sequence continues indefinitely. However, it may happen that 0 turns up, in which case you can proceed no further.

Here are some questions:

A.1. To get a sense of the generic behaviour of these sequences, begin with a pair of starting values and see what happens.

A.2. How does the sequences whose first two terms are a, b compare with that obtained by taking the terms in the oppositve order b, a?

A.3. What are forbidden terms who presence may lead at some point to the value 0 in the sequence?

A.4. What are the sequences all of whose terms are integers?

A.5. Which sequences are symmetrical in the sense that they are the same when read backwards?

2. The sequence $\mathfrak{B}$.

This time, let us start with three numbers a, b and c, and make them the first three terms of a sequence a, b, c, d, e, f each of whose terms in the sum of the two previous terms plus 1 divided by the third previous term:

$$d = \frac{b+c+1}{a}, \qquad e = \frac{c+d+1}{b}, \qquad f = \frac{d+e+1}{c}$$

and so on. More formally, we can define the sequence bilaterally by

$$x_{n+1} = \frac{x_n + x_{n-1} + 1}{x_{n-2}},$$

where n ranges over all the integers. As before, if we arrive at the value 0, the sequence stops, so we need to prevent this from occurring.

B.1. Try various triples of starting values and see what happens.

B.2. What happens if you take the starting values a, b, c in the order c, b, a?

B.3. What restrictions should be placed on the starting values a, b, c to prevent the term 0 from appearing?

B.4. What are the sequences all of whose terms are integers?

B.5. What happens if the first three terms are consecutive integers in either increasing or decreasing order?

B.6. Which sequences are symmetrical in the sense that they are the same when read backwards?

3. The sequence $\mathfrak{C}$.

In the same vein, we can begin with four numbers a, b, c, d and continue the sequence by adding three consecutive terms plus 1 and dividing by the previous term:

$$e = \frac{b+c+d+1}{a}$$

and so on. More formally, we have

$$x_{n+1} = \frac{x_n + x_{n-1} + x_{n-2} + 1}{x_{n-3}},$$

where n ranges over all the integers. Once again, if we arrive at a 0 term, then we must stop.

C.1. Experiment with various quadruples of starting values.

C.2. What happens if you take the starting values in opposite order?

C.3. What restrictions should be placed on the starting values in order to avoid arriving at a term equal to 0?

C.4. Are there any sequences all of whose terms are integers?

C.5. Classify sequences as being symmetrical (the same when read backwards) and non-symmetrical.

At this point, you may wish to stop reading, explore the situation more, and perhaps formulate some questions of your own.

4. The sequence $\mathfrak{A}$.

In every case, unless the sequence gets 0 at some point, it has period 5, *i.e.*, it repeats itself over and over after the first five terms. This can be checked algebraically; we have the period

$$\left(a, \quad b, \quad \frac{b+1}{a}, \quad \frac{a+b+1}{ab}, \quad \frac{a+1}{b}\right)$$

after which the terms continue $a, b, \ldots$. If we start with b, a, then we can just read this sequence in the opposite direction.

In particular, in examining possible sequences, we need only consider only those for which $|a| \leq |b|$. (For the other case, we only have to read the sequences from right to left.)

If we wish to have integer entries, then a and b must be integers and the denominator of each fraction in the foregoing list must divide the numerator. If we assume that $|a| \leq |b|$, then, since $|b|$ is a divisor or $|a+1|$,

$$|a| \leq |b| \leq |a+1|.$$

This forces a to be positive and b to take one of the four values a, $a+1$, $-a$, $-(a+1)$. Look at these possibilities in turn:

If $b = a$, then the sequence begins $a, a, (a+1)/a$. The only way that a can divide $a+1$ is for $a = 1$, and we obtain the sequence

$$1, 1, 2, 3, 2, 1, 1, 2, 3, 2, 1, 1, 2, 3, 2, \ldots.$$

If $b = a+1$, then the sequence begins $a, a+1, (a+2)/a$ so a must be either 1 or 2. In this case we get either the period

$$(1, 2, 3, 2, 1)$$

or the period

$$(2, 3, 2, 1, 1),$$

which gives the same sequence with a different starting point.

If $b = -a$, then the sequence begins $a, -a, -(a-1)/a$, so $a = 1$. But this leads to 0 and is impossible.

Finally, if $b = -a-1$, then the sequence begins $a, -a-1, -1, 0$, so this case is impossible.

4. The sequence $\mathfrak{B}$.

In this situation, we find that unless 0 appears, the sequence has period 8. If we start with a, b, c, we obtain the period

$$\left(a, \quad b, \quad c, \quad \frac{b+c+1}{a}. \quad \frac{a+b+c+1+ac}{ab}, \right.$$
$$\left. \frac{(a+b+1)(b+c+1)}{abc}, \quad \frac{a+b+c+1+ac}{bc}, \quad \frac{a+b+1}{c}\right).$$

Note that we can read this sequence in the opposite direction and it will still satisfy the rule of formation.

B.7. There is the possibility that the sequence $\mathfrak{B}$ might have a shorter period. The possibilities for the length of the period are 1, 2 and 4. Determine all the possible periodic sequences.

Let us investigate when all the entries are integers; this requires that only the numbers in each period are integers. To begin with, we suppose that all the entries are positive and that a is a maximum entry. Then $0 < b+c+1 \leq 2a+1$. It follows from this that the fourth term of the period, $(b+c+1)/a$, being an integer, does not exceed 2. Hence, either $b+c = 2a-1$ or $b+c = a-1$.

In the first instance, the only possibilities are $(b, c) = (a, a-1)$ or $(b, c) = (a-1, a)$. The fifth term is respectively $1 + (2/a)$ or $1 + (4/(a-1))$ so that a is one of 2, 3, 5. Checking these out leads to sequences with one of these periods: $(5, 4, 5, 2, 2, 1, 2, 2)$ and $(3, 2, 3, 2, 3, 2, 3, 2)$

Otherwise $b + c = a - 1$, and the period is

$$(a, b, a-b-1, 1, \frac{a+1}{b} - 1, \frac{a+b+1}{b(a-b-1)} = \frac{1}{b}\left(1 + \frac{2(b+1)}{a-(b+1)}\right), \dots).$$

Since the sixth term is an integer, we must have $ab - b^2 - b \leq a + b + 1$, or

$$a \leq \frac{(b+1)^2}{b-1} = b + 3 + \frac{4}{b-1}.$$

When $b = 2$, a must be odd and $a - 3$ must divide 6. Hence $a = 5$ or $a = 9$ and we obtain the periods: $(5, 2, 2, 1, 2, 2, 5, 4)$ and $(9, 2, 6, 1, 4, 1, 6, 2)$.

When $b = 3$, $a - 2$ must be a multiple of 3 and $a - 4$ must divide 8. Hence $a = 5, 8$ and we obtain the periods: $(5, 3, 1, 1, 1, 3, 5, 9)$ and $(8, 3, 4, 1, 2, 1, 4, 3)$.

When $b = 4$, a must be either 7 or 15, but both these fail on the sixth term.

When $b = 5$, then $a = 9$ and we obtain $(9, 5, 3, 1, 1, 1, 3, 5)$.

If $b \geq 6$, then $4/(b-1) < 1$ and so $a \leq b+3$. Hence $a-3 \leq b \leq a$. Since $b+c = a-1$ and $c \geq 1$, we must have $b = a-2$ or $b = a-1$. We are led to the periodic sequences:

$$(a, a-2, 1, 1, \frac{3}{a-2}, \dots) \quad \text{and} \quad (a, a-3, 2, 1, \frac{4}{a-3}).$$

Since $a-2 \geq 6$, $a \geq 8$ and the fifth term in not an integer. Thus, this case is not possible.

Consider periods that have at least one negative number. If the sequence has three negative terms in a row, the next term must be positive. Thus, the period must have at least one positive entry. Therefore, wolog, we assume that $a \geq 1 > -1 \geq b$.

First suppose that $c > 0$; we can also suppose that $c \leq a$, since reversing the sequence gives a valid sequence. Let $b+c+1 > 0$. Since $b+c+1$ is divisible by a, then

$$a \leq b+c+1 = c+(b+1) \leq c \leq a,$$

from which $a = c$ and $b = -1$. This leads to the period

$$(a, -1, a, 1, -(a+2), -1, -(a+2), 1)$$

for $a \neq 0, -2$.

The other possibility is that

$$a \geq c \geq 1 > -1 \geq b+c+1 > b.$$

Since a divides $b+c+1$, then $b+c+1 = -ka$ where k is a positive integer. Therefore $b = -ka-c-1$ and we are led to the period

$$(a, -ka-c-1, c, -k, -\frac{c+1-k}{c+1+ka} \dots).$$

The denominator of the fraction in the fifth entry is positive and exceeds the numerator. so if the fifth entry is an integer, $k-c-1$ must be positive and at least as large as $c+1+ka$. But this would imply that $k-c-1 \geq c+1+ka$ or $0 \geq 2(c+1)+k(a-1)$, an impossibility.

If $a > 0 > -1 \geq b, c$, then $b+c+1 < 0$, and the fourth term is negative. But then the fifth term in the period must be positive. Noting that we cannot have two negative terms immediately preceded and followd by positive terms, we have these possibilities for the signs of the terms in the period:

$$(+, -, -, -, +, +, -, +)$$
$$(+, -, -, -, +, -, +, ?)$$
$$(+, -, -, -, +, -, -, -)$$

Since the first and second of these involve the subsequence $+, -, +$, which can be placed at the front, these cases have been considered. Only the third possibility remains to be considered.

Suppose that the period is $(p, -q, -r, -s, t, -u, -v, -w)$ with all of p, q, r, s, t, u, v, w positive. Then $p-q+1 = rw$ and $-q-r+1 = -ps$, whence $p+r = rw+ps$ or $p(s-1)+r(w-1) = 0$. Hence $w = s = 1$. Also $p-w+1 = qv$ and $-v-w+1 = -pu$, whence $p(u-1)+v(q-1)=0$. Hence $u=q=1$. It follows that $p=r=v$ and we are led to

$$(p,-1,-p,-1,p,-1,-p,-1).$$

In summary, we have the following possible periods all of whose entries are integers:

$$(9,5,3,1,1,1,3,5)$$
$$(9,2,6,1,4,1,6,2)$$
$$(8,3,4,1,2,1,4,3)$$
$$(5,4,5,2,2,1,2,2)$$
$$(3,2,3,2,3,2,3,2)$$
$$(a,-1,a,1,-(a+2),-1,-(a+2),1) \quad (a \neq 0,-2)$$
$$(p,-1,-p,-1,p,-1,-p,-1) \quad (p \neq 0).$$

Note that all of the sequences obtained with these periods are symmetrical, *i.e.*, they are the same when they are read backwards.

If the first three terms are consecutive integers $b-1$, b, $b+1$, then the terms are

$$b-1, \qquad b, \qquad b+1, \qquad \frac{2(b+1)}{b-1}, \qquad \frac{b+3}{b-1}.$$
$$\frac{4}{b-1}, \qquad \frac{b+3}{b+1}, \qquad \frac{2(b-1)}{b+1}, \qquad b-1, \qquad \ldots.$$

Note that the consecutive pairs of the fourth, fifth and sixth terms differ by 1.

5. The sequence $\mathfrak{C}$.

For the sequence $\mathfrak{C}$, there seems to be no periodcity for all sequences. It is possible to have quite long chains of integers; here are some maximal subsequences of integers:

$$\{\ldots,-1,-2,-1,5,-3,-1,-2,-1,1,1,-1,-2,-1,-3,5,-1,-2,-1,\ldots\} \quad \text{(18 terms)}$$
$$\{\ldots,-4,-3,-1,-5,2,1,1,-1,1,2,3,-7,-1,-2,-3,\ldots\} \quad \text{(15 terms; non}-\text{symmetric)}$$
$$\{\ldots,16,17,9,5,2,1,1,1,2,5,9,17,16,\ldots\} \quad \text{(13 terms; all positive)}$$
$$\{\ldots,5,3,4,2,2,3,2,4,5,4,7,\ldots\}$$
$$\{\ldots,7,8,3,2,2,1,2,2,3,8,7,\ldots\}$$
$$\{\ldots,4,5,3,3,3,2,3,3,3,5,4,\ldots\}$$

C.6. If we want to find a sequence all of whose entries are integers, we might consider a periodic sequence. At this point, all positive integers are available as possible lengths of a period. Consider the possibility of periods of length 1, 2, 3, 4, *etc.*.

6. Invariant functions.

For each sequence, we can define an associated function. For the sequence $\mathfrak{A}$, In the case of the period 5 sequence, we can construct a transformation in the plane which takes each pair of consecutive terms to the next pair of consecutive terms: $T_A(x,y) = (y, (y+1)/x)$.

A.6. Determine a function $h(x,y)$ which is invariant under this transformation, *i.e.* $h(T(x,y)) = h(y,(y+1)/x) = h(x,y)$.

We can do the same for the sequence $\mathfrak{B}$: $T_B(x,y,z) = (y,z,(y+z+1)/x)$.

B.8. Determine a function $h(x,y,z)$ which is invariant under this transformation.

Likewise, for the sequence $\mathfrak{C}$, we can define: $T_C(x,y,z,w) = (y,z,w,(y+z+w+1)/x)$.

C.7. Determine a function $h(x,y,z,w)$ which is invariant under this transformation.

7. Periodic sequences.

The only constant sequences $\mathfrak{A}$ has each term equal to $\frac{1}{2}(1 \pm \sqrt{5})$, one of the roots of the quadratic function $t^2 - t - 1$.

The only constant sequences $\mathfrak{B}$ has each term equal to $1 \pm \sqrt{2}$, one of the roots of the quadratic function $t^2 - 2t - 1$.

For a sequence $\mathfrak{B}$ with a period (a,b) of length 2, we require that $ab = a+b+1$, so that $(a-1)(b-1) = 2$. We obtain the possibilities $(a,b) = (1+u, 1+2/u)$ for the period, where $u \neq 1, 2$. An integer example is $(2,3)$.

For a sequence $\mathfrak{B}$ with a period (a,b,c,d) of length 4, we require that $b+c+1 = da$, $c+d+1 = ab$, $d+a+1 = bc$ and $a+b+1 = cd$. Subtracting adjacent pairs of these equations leads to

$$(b-d)(a+1) = (c-a)(b+1) = (d-b)(c+1) = (c-a)(d+1) = 0.$$

Also, we have that $ad + bc = a+b+c+d+2 = ab+cd$, whence $(a-c)(b-d) = 0$.

The case $a = c$ and $b = d$ takes us back to the period of length 2, so we may suppose that $a \neq c$. Then we have $b = d = -1$, from which $c = -a$ and we are led to the period $(a, -1, -a, -1)$ for any nonzero real a.

For the sequence $\mathfrak{C}$, the constant sequence has terms equal to one of $\frac{1}{2}(3 \pm \sqrt{13})$, the roots of $t^2 - 3t - 1$.

For period 2, we require that $2a + b + 1 = b^2$ and $a + 2b + 1 = a^2$, which leads to $(a-b)(a+b+1) = 0$. If the sequence is nonconstant, then $a = b^2$ and $b = a^2$, so that $(a,b) = (\omega, \omega^2), (\omega^2, \omega)$ where ω is an imaginary cube root of unity, i.e. a root of $t^2 + t + 1$.

If (c,d,a,b,c,d,a,b) is the period of a period 4 sequence, then $b+c+1=ad$, $a+b+1=cd$, $c+d+1=ab$ and $a+d+1=bc$. These lead to $(a-c)(1+d)=0$ and $(a-c)(1+b)=0$.

If $a=c$, then we must have $a+b+1=ad$ and $a+d+1=ab$, whence $(b-d)(1+a)=0$. The case $b=d$ leads to a period 2 sequence. If $a=-1$, then $d=-b$. This leads to the period 4 $(-1,b,-1,-b)$. The other possibility is that $b=d=-1$, which leads to $c=-a$ and the period 4 $(a,-1,-a,-1)$, which is essentially the same as before.

For period 5 sequences with the period (a,b,c,d,e), we require that $a+b+c+1=de$, $b+c+d+1=ea$, $c+d+e+1=ab$, $d+e+a+1=bc$, $e+a+b+1=cd$, from which $(a-d)(e+1)=(b-e)(a+1)=(c-a)(b+1)=(d-b)(c+1)=(e-c)(d+1)=0$.

Wolog, we may suppose that $a\neq d$. Then $e=-1$, so that $c+d=ab$, $a+d=bc$ and $a+b=cd$. Also $a+b+c+d+1=a+ea=a-a=0$. Therefore

$$(c+1)(d+1)=cd+c+d+1=a+b+c+d+1=0,$$

and either $c=e=-1$ or $d=e=-1$.

In the first case,

$$(a+1)(b+1)=ab+a+b+1=a+b+c+d+1=0,$$

so that $a=-1$ or $b=-1$. If $a=c=e=-1$, then $b+d=1$ and we obtain the period $(-1,b,-1,1-b,-1)$ which contains only integers when b is an integer not equal to 0 or 1. $b=c=e=-1$ leads essentially to the same conclusion.

In the second case, $d=e=-1$, whereupon $a+b+c=de-1=0$, $c-1=ab$ and $a-1=bc$. Thus $(a-c)(b+1)=0$. However, neither of these work.

8. Invariant functions.

To find an invariant function $h(x,y)$ for the sequence $\mathfrak{A}$, we can simply add together the five terms of the period to get

$$h(x,y)=x+y+(y+1)/x+(x+y+1)/xy+(x+1)/y=\frac{x^2y+xy^2+x^2+y^2+2(x+y)+1}{xy}$$

which remains invariant under the transformation T_A. Alternatively, we can multiply the five terms of the period to obtain

$$\frac{(x+1)(y+1)(x+y+1)}{xy}=h(x,y)+3.$$

In the case of the sequence $\mathfrak{B}$, we have the related transformation $T(x,y,z)=(y,z,(y+z+1)/x$ in 3-space with invariant function

$$h(x,y,z)=(xyz)^{-1}[xyz(x+y+z)+(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2)$$
$$+(x^2+y^2+z^2)+3(xy+yz+zx)+2(x+y+z)+1].$$