Commuting exponentially.

A mathematical vignette

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0. The investigation.

If x and y are two positive real numbers, when is it possible that $x^y = y^x$? This is clearly true when $x = y$, so we can exclude this case from further discussion. This is a question that can be tackled by pupils at various levels. For elementary students, this can simply be posed for positive integers. For secondary students, we can extend the discussion to rational numbers, and ask in particular, whether there are infinitely many rational pairs for which the equation holds. For students who have studied calculus, we can move to the real of positive real numbers.

1. Positive integer solutions.

It is reasonable for some groups of pupils to discover by trial and error that $2^4 = 16 = 4^2$. Suppose that $x < y$. What can x be? It cannot be 1. If $x = 2$, is $y = 4$ the only possibility? We can check that $2^3 < 3^2$, while $2^5 = 32 > 25 = 5^2$, $2^6 = 64 > 36 = 6^2$. It seems that for larger values of y, 2^y is bigger than y^2 and the gap between them increases. Is there any way for an elementary pupil familiar with the laws of indices to see this?

It would be interesting to see what students come up with. When $y = 2z$ is an even integer, then $2^y = (2^z)^2$ and it becomes a matter of checking whether $2^z > 2z$ or $2^{z-1} > z$.

Now look at larger values of x; is there a systemic inequality between x^y and y^x. For example $3^6 = 9^3$ is bigger than 6^3.

2. Positive rational solutions.

The first issue to settle is that, for each fixed positive value of $x > 1$, x^y is a function that increases strictly from 1 at $y = 0$ and becomes arbitrarily large. For each fixed positive value of $x < 1$, x^y decreases strictly from 1 at $y = 0$ to zero.

If $1 < x, y$, we can write $y = x^z$ some positive value of z. This has the effect of providing the same base for the exponentials on both sides of the equation:

$$x^{x^z} = x^y = y^x = (x^z)^x = x^{zx}.$$

Therefore $x^z = xz$ from which $x^{z-1} = z$. This can be solved for x, and then for y:

$$x = z^{\frac{1}{z-1}}, \qquad y = z^{\frac{z}{z-1}}.$$

It is a useful exercise in exponents to check that these values of x and y achieve $x^y = y^x$, at first for specific values of z such as $z = 2$ and $z = 3$, and then generally. The case $z = 3$ will get us into surds, and in general, values of z will lead to noninteger exponents. What values of z will reproduce the known cases $(x, y) = (2, 4)$ and $(x, y) = (4, 2)$.

For more thoughtful students, it would be interesting to reflect on where the case $x = y$ fits into this. This would apparently correspond to $z = 1$. However, using this value to recover x and y leads to undefined expressions.

Since $y = x^z$ is equivalent to $x = y^{1/z}$, it should happen that replacing z by $1/z$ in the formulas for x and y should give their values in the opposite order. Check that this in fact does occur. What value of z yields the pair $(x, y) = (4, 2)$?

Are $(x, y) = (2, 4)$ and $(x, y) = (2, 4)$ the only solutions where x and y are both rational? One attempt to get one such pair is to select z so that z is rational and $1/(z - 1)$ is a rational integer. If we take $z = 2/3$, we find that

$$(x, y) = ((2/3)^{-3}, (2/3)^{-2}) = (27/8, 9/4).$$

If we take $z = 4/3$, then

$$(x, y) = ((4/3)^3, (4/3)^4) = (64/27, 256, 81).$$

Check out the situation with $z = (k \pm 1)/k$. Do this give all the rational solutions?

It remains to check out whether solutions exist when $x < 1$. What happens if $x < 1 < y$? if $x, y < 1$? In the second case, set $x = 1/u$ and $y = 1/v$. When students have learned some calculus, we can get further insight on when solutions exist.

3. Analysis.

The equation $u^v = v^u$ is equivalent to

$$\frac{\log u}{u} = \frac{\log v}{v}.$$

Thus, we need to examine values of x where the function $f(x) = (\log x)/x$ assumes the same value twice. The graph of $y = f(x)$ is readily sketched; the function $f(x)$ increases on the interval $(0, e)$ and decreases on (e, ∞). It assumes each value in the open interval $(0, 1/e)$ exactly twice at points u and v for which $1 < u < e < v$.

The relationship between u and v where $f(u) = f(v)$ permits us to sketch the graph of the equation $x^y = y^x$, and note that it consists of two curves, one the graph of $y = x$ and the other a hyperbola-shaped curve that intersects it at (e, e).

Further investigations might consider when the equation $y^x = x^y$ makes sense for nonpositive values of x and y, and, if so, what solutions there might be. And then of course, we can move the situation into the complex realm.