Can you hear the shape of a drum?

Fall 2018

APM384 Partial Differential Equations

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Wave Equation

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1. What do we hear when someone plucks this string?

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1. What do we hear when someone plucks this string?
2. If we had perfect hearing, could we tell the length L of the string?

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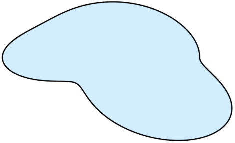
Wave Equation

$$\begin{cases} u_{tt} = c^2 \Delta u, & x \in \Omega \quad , \quad t > 0 \\ u = 0, & x \in \partial\Omega \end{cases}$$



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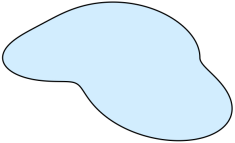

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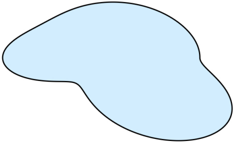

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where λ_n and φ_n are eigenvalues and eigenfunctions of

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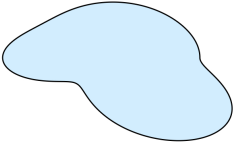

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4 What determines λ_n?

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Weyl 1911. Area of drumhead is determined by eigenvalues:

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- Perimeter is also determined by eigenvalues
- If the domain is multiply-connected, the eigenvalues indicate the number of holes.

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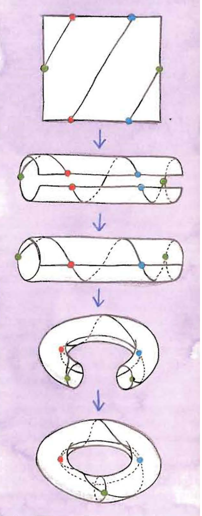

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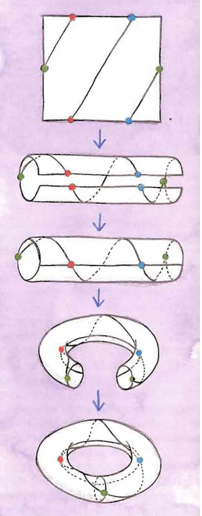

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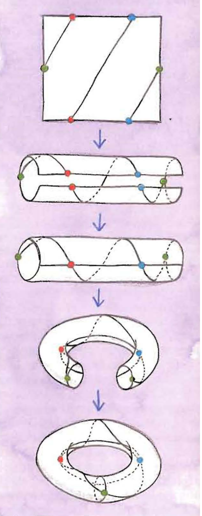

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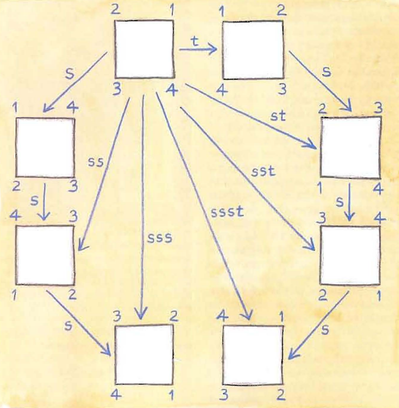
Sunada 1984. Group Theory → Manifolds



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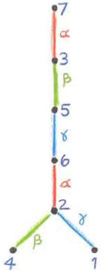

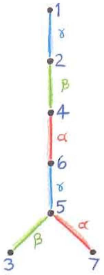

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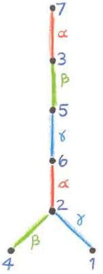

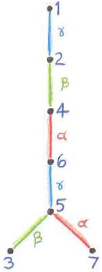

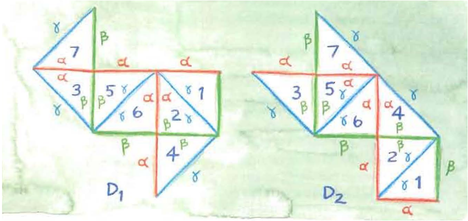

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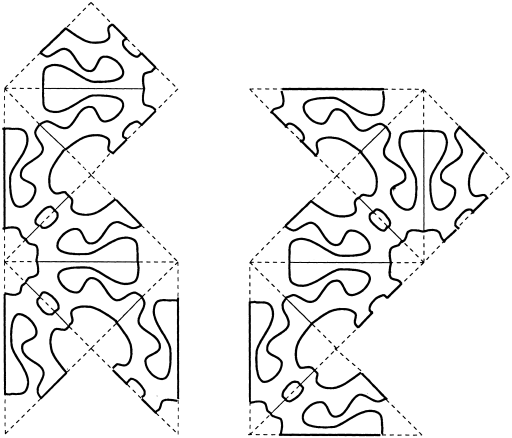

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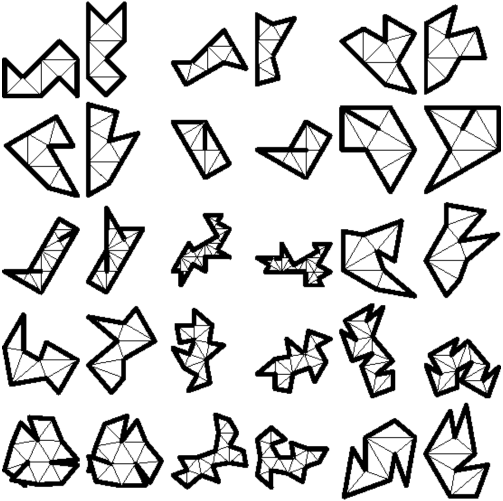

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Even though the problem has been solved, there is still ongoing research.

- It is known that if the domain has the same eigenvalues as a disk, then it must be a disk.

Zelditch 2000. Showed that there is a class of convex set (with analytic boundary and two axes of symmetry) for which the eigenvalues of the Laplacian determine the domain.