University of Toronto

Faculty of Applied Science and Engineering

Solutions to Final Examination, December 2017

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT186H1F - Calculus I

Examiners: S. Amelotte, F. Bischoff, D. Burbulla, S. Cohen, J. Enns, M. Matviichuk, M. Palasciano, K. Pham

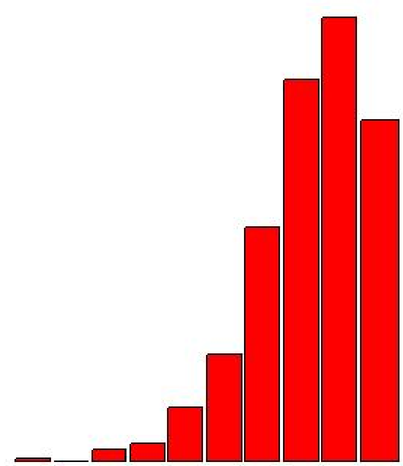
Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

- In Question 1(c) very few students knew, or bothered, to check that y' has an infinite limit as $x \to 0$.
- Many students tried integration by parts for Questions 4(a) and 5(b), which we had not covered, and which was *not* necessary
- In Question 5, some students used Newton's Method to approximate the intersection point of the two curves, but this was not necessary, as you can find the intersection point by "inspection," by using some basic trig values, i.e. $\tan(\pi/3) = \sqrt{3}$.

Breakdown of Results: 731 registered students wrote this test. The marks ranged from 5% to 100%, and the average was 77.3%. There were 13 perfect papers. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	23.2%
A	50.4%	80-89%	27.2%
B	23.4%	70-79%	23.4%
C	14.4%	60-69%	14.4%
D	6.6%	50-59%	6.6%
F	5.2%	40-49%	3.3%
		30-39%	1.1%
		20-29%	0.7%
		10-19%	0.0%
		0-9%	0.1%



1. [avg: 7.8/10] Find the following:

(a) [3 marks] $\lim_{x\to 0} \frac{\sin(2x)}{3x}$

Solution: the limit is in the 0/0 form so use L'Hopital's Rule.

$$\lim_{x\to 0} \frac{\sin(2x)}{3x} = \lim_{x\to 0} \frac{2\cos(2x)}{3} = \frac{2}{3}$$

(b) [4 marks] the value of $\frac{dy}{dx}$ at the point $(x, y) = (1, 2)$ if $x^2 + xy^3 = x + 8$.

Solution: differentiate implicitly.

$$x^2 + xy^3 = x + 8 \Rightarrow 2x + 1 \cdot y^3 + 3xy^2 \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1 - 2x - y^3}{3xy^2};$$

so at the point $(x, y) = (1, 2)$,

$$\frac{dy}{dx} = \frac{1 - 2 - 8}{3 \cdot 4} = -\frac{9}{12} = -\frac{3}{4}.$$

(c) [3 marks] the equation of the vertical tangent line to the graph of $y = x^{7/3} - 7\,x^{1/3}$

Solution: let $y = f(x)$. We need to find at which point $f'(x)$ is undefined.

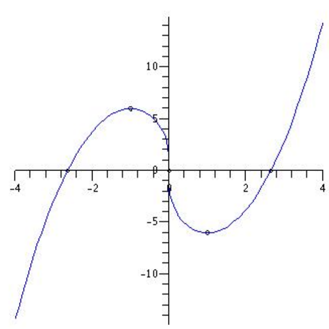
$$f'(x) = \frac{7x^{4/3}}{3} - \frac{7}{3x^{2/3}};$$

so $f'(0)$ is undefined. To confirm there is a vertical tangent at $x = 0$ you have to check that there is an infinite limit of $f'(x)$ as $x \to 0$:

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(\frac{7x^{4/3}}{3} - \frac{7}{3x^{2/3}} \right) = \lim_{x\to 0} \frac{7}{3} \left(\frac{x^2 - 1}{x^{2/3}} \right) = -\infty.$$

So, yes, the vertical line with equation $x = 0$ is a vertical tangent line to the graph of $y = f(x)$.

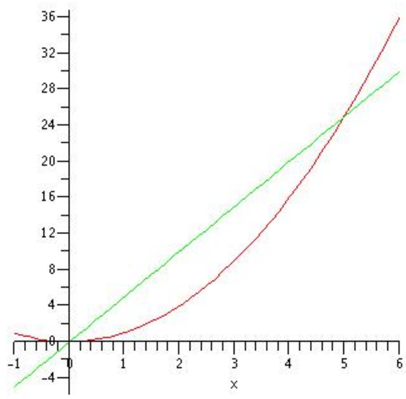
Aside: this is the same function that appeared on test 2; its graph is to the right:



2. [avg: 9.0/10] Let R be the region bounded by the curves $y = 5\,x$ and $y = x^2$ for $0 \le x \le 5$. Find the following:

(a) [5 marks] the volume of the solid generated by revolving R about the x-axis.

Solution: on the interval $[0, 5]$ the line $y = 5x$ is above the parabola $y = x^2$. (See diagram.) So, using the method of discs/washers, the volume is



$$\begin{aligned} V &= \int_0^5 \pi((5x)^2 - (x^2)^2)\,dx \\ &= \pi \int_0^5 (25x^2 - x^4)\,dx \\ &= \pi \left[\frac{25x^3}{3} - \frac{x^5}{5}\right]_0^5 \\ &= 625\pi\left(\frac{5}{3} - 1\right) = \frac{1250\,\pi}{3} \approx 1309 \end{aligned}$$

Alternate Solution: use the method of shells and integrate with respect to y.

$$V = \int_0^{25} 2\pi\,y\left(\sqrt{y} - \frac{y}{5}\right)\,dy = 2\pi\left[\frac{2y^{5/2}}{5} - \frac{y^3}{15}\right]_0^{25} = \frac{1250\,\pi}{3}$$

(b) [5 marks] the volume of the solid generated by revolving R about the line $x = -3$.

Solution: use the method of shells.

$$\begin{aligned} V &= \int_0^5 2\pi(x+3)(5x - x^2)\,dx \\ &= 2\pi \int_0^5 (2x^2 + 15x - x^3)\,dx \\ &= 2\pi\left[\frac{2x^3}{3} + \frac{15x^2}{2} - \frac{x^4}{4}\right]_0^5 \\ &= 250\pi\left(\frac{2}{3} + \frac{3}{2} - \frac{5}{4}\right) \\ &= \frac{1375\,\pi}{6} \approx 720 \end{aligned}$$

Alternate Solution: use the method of washers and integrate with respect to y. Then $V =$

$$\int_0^{25} \pi\left((\sqrt{y}+3)^2 - \left(\frac{y}{5}+3\right)^2\right)\,dy = \pi\int_0^{25}\left(-\frac{y}{5} + 6\sqrt{y} - \frac{y^2}{25}\right)\,dy = \pi\left[-\frac{y^2}{10} + 4y^{3/2} - \frac{y^3}{75}\right]_0^{25} = \frac{1375\pi}{6}$$

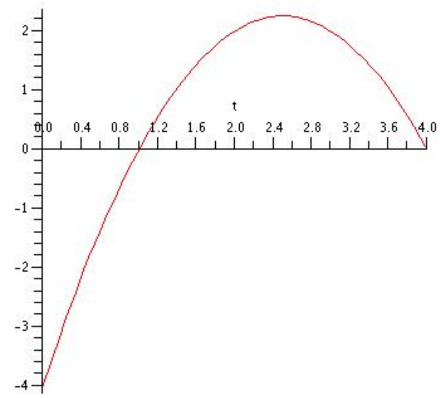
3. [avg: 8.95/10] Let $v = -t^2 + 5t - 4$ be the velocity of a particle at time t, for $0 \le t \le 4$. Find:

(a) [4 marks] the average velocity of the particle.

Solution: direct application of average formula.

$$v_{avg} = \frac{1}{4-0}\int_0^4 v\,dt = \frac{1}{4}\int_0^4(-t^2+5t-4)\,dt = \frac{1}{4}\left[-\frac{t^3}{3}+\frac{5t^2}{2}-4t\right]_0^4 = \frac{1}{4}\left(-\frac{64}{3}+40-16\right) = \frac{2}{3}$$

(b) [6 marks] the average speed of the particle.



Solution: the average speed is given by

$$\frac{1}{4}\int_0^4 |v|\,dt.$$

Since v changes signs on the interval $[0,4]$ we have to calculate two separate integrals:

$$\frac{1}{4}\int_0^4 |v|\,dt = \frac{1}{4}\int_0^1(-v)\,dt + \frac{1}{4}\int_1^4 v\,dt$$

So the average speed is

$$\begin{aligned}
&\frac{1}{4}\int_0^1 (t^2-5t+4)\,dt + \frac{1}{4}\int_1^4(-t^2+5t-4)\,dt \\
=\;& \frac{1}{4}\left[\frac{t^3}{3}-\frac{5t^2}{2}+4t\right]_0^1 + \frac{1}{4}\left[-\frac{t^3}{3}+\frac{5t^2}{2}-4t\right]_1^4 \\
=\;& \frac{1}{4}\left(\frac{1}{3}-\frac{5}{2}+4\right) + \frac{1}{4}\left(-\frac{64}{3}+40-16+\frac{1}{3}-\frac{5}{2}+4\right) \\
=\;& \frac{11}{24}+\frac{2}{3}+\frac{11}{24} = \frac{19}{12}
\end{aligned}$$

Aside: note that the difference of the above two integrals gives the average value:

$$-\frac{11}{24}+\frac{2}{3}+\frac{11}{24} = \frac{2}{3}.$$

4. [avg: 7.1/10] Find the exact values of the following:

(a) [3 marks] $\int_{-3}^{3} (x^7 \cos x - x^2)\, dx$

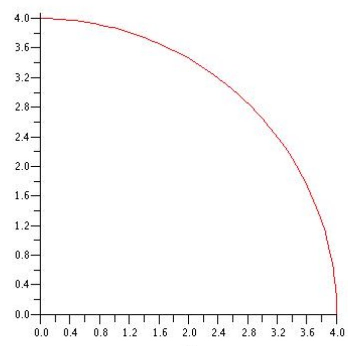
Solution: use symmetry to simplify the calculations.

$$\begin{aligned}\int_{-3}^{3} (x^7 \cos x - x^2)\, dx &= \int_{-3}^{3} \underbrace{x^7 \cos x}_{odd}\, dx - \int_{-3}^{3} \underbrace{x^2}_{even}\, dx \\ &= 0 - 2\int_0^3 x^2\, dx \\ &= -2\left[\frac{x^3}{3}\right]_0^3 = -18\end{aligned}$$

(b) [3 marks] $\int_0^4 \sqrt{16 - x^2}\, dx$ (Hint: interpret the integral as an area.)

Solution: $y = \sqrt{16 - x^2} \Rightarrow x^2 + y^2 = 16$. so the integral represents a quarter of the area of a circle of radius 4. (See diagram.) Therefore

$$\int_0^4 \sqrt{16 - x^2}\, dx = \frac{1}{4}(16\pi) = 4\pi.$$



(c) [4 marks] $\int_0^4 x\sqrt{16 - x^2}\, dx$

Solution: use a substitution. Let $u = 16 - x^2$. Then $du = -2x\, dx$ and

$$\begin{aligned}\int_0^4 x\sqrt{16 - x^2}\, dx &= -\frac{1}{2}\int_{16}^{0} \sqrt{u}\, du \\ &= \frac{1}{2}\int_0^{16} \sqrt{u}\, du \\ &= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^{16} \\ &= \frac{1}{3}(16^{3/2} - 0) \\ &= \frac{64}{3}\end{aligned}$$

5. [avg: 7.0/10] Let A be the area of the region bounded by $y = \tan^{-1} x$ and $y = \dfrac{\pi\, x}{3\sqrt{3}}$. (Draw a diagram.)

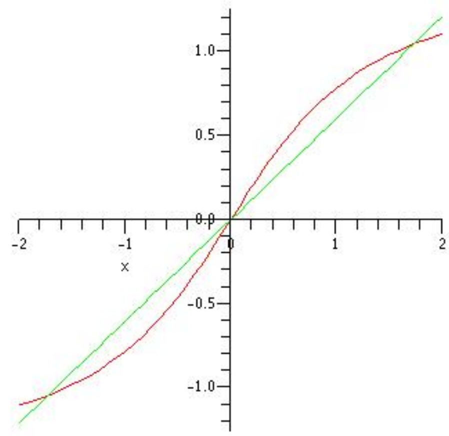
(a) [3 marks] Express the value of A in terms of one or more integrals with respect to x.

Solution: since

$$\tan^{-1}\sqrt{3} = \frac{\pi}{3} \text{ and } \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3},$$

the intersection points of the two graphs are $(0,0)$, $\left(\sqrt{3}, \frac{\pi}{3}\right)$ and $\left(-\sqrt{3}, -\frac{\pi}{3}\right)$. With respect to x the value of A is

$$2\int_0^{\sqrt{3}} \left(\tan^{-1} x - \frac{\pi\, x}{3\sqrt{3}}\right) dx$$



(b) [3 marks] Express the value of A in terms of one or more integrals with respect to y.

Solution: with respect to y the value of A is

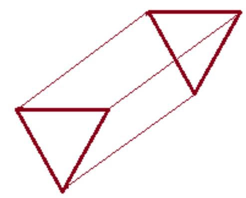
$$2\int_0^{\pi/3} \left(\frac{3\sqrt{3}\, y}{\pi} - \tan y\right) dy$$

(c) [4 marks] Find the value of A.

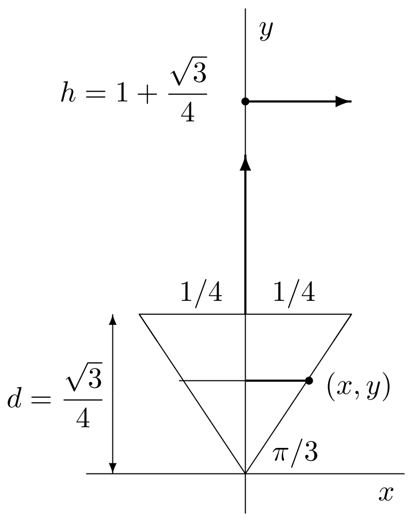
Solution: integrate with respect to y, since we don't know an antiderivative of $\tan^{-1} x$.

$$\begin{aligned}
A &= 2\int_0^{\pi/3} \left(\frac{3\sqrt{3}\, y}{\pi} - \tan y\right) dy \\
&= \frac{6\sqrt{3}}{\pi}\int_0^{\pi/3} y\, dy - 2\int_0^{\pi/3} \frac{\sin y}{\cos y}\, dy \\
&= \frac{6\sqrt{3}}{\pi}\left[\frac{y^2}{2}\right]_0^{\pi/3} + 2\int_1^{1/2} \frac{du}{u}. \text{ with } u = \cos y \\
&= \frac{6\sqrt{3}}{\pi}\left(\frac{\pi^2}{18}\right) + 2(\ln(1/2) - \ln 1) \\
&= \frac{\pi}{\sqrt{3}} - 2\ln 2, \text{ or } \frac{\pi}{\sqrt{3}} - \ln 4 \approx 0.4275
\end{aligned}$$

6. [avg: 7.9/10] A water trough has a horizontal length of 2 m, and vertical cross sections which are equilateral triangles, with each side of length 0.5 m . (For the shape of the trough, see the diagram to the right.) If the trough is full, how much work is required to pump all the water up to an exit pipe 1 m above the top of the trough? (Assume the density of water is $\rho = 1000$ kg/m^3 and that $g = 9.8$ m/sec^2.)



Solution: consider a vertical cross-section:



To calculate the vertical depth of the tank you can use the Pythagorean Theorem:

$$d^2 + (1/4)^2 = (1/2)^2 \Rightarrow d^2 = \frac{3}{16} \Rightarrow d = \frac{\sqrt{3}}{4}.$$

To calculate the width of the tank at height y there are two approaches you can use:

1. $y = \tan(\pi/3)\, x = \sqrt{3}\, x \Rightarrow x = \dfrac{y}{\sqrt{3}}$.
2. or, by similar triangles,

$$\frac{x}{y} = \frac{1/4}{\sqrt{3}/4} = \frac{1}{\sqrt{3}} \Rightarrow x = \frac{y}{\sqrt{3}}.$$

.

Either way, the horizontal cross-sectional area of the tank at height y is given by

$$A(y) = \text{ length } \times \text{ width } = 2\,(2x) = 4x = \frac{4\,y}{\sqrt{3}},$$

and the work required to empty the tank, in Joules, is given by

$$\begin{aligned}
W &= \int_0^d \rho\, g\, A(y)(h - y)\, dy \\
&= \int_0^{\sqrt{3}/4} \rho\, g\, \frac{4\,y}{\sqrt{3}}\, (1 + \frac{\sqrt{3}}{4} - y)\, dy \\
&= \frac{\rho\, g}{\sqrt{3}} \int_0^{\sqrt{3}/4} \left(4y - 4y^2\right)\, dy + \rho\, g \int_0^{\sqrt{3}/4} y\, dy \\
&= \frac{\rho\, g}{\sqrt{3}} \left[2y^2 - \frac{4y^3}{3}\right]_0^{\sqrt{3}/4} + \rho\, g \left[\frac{y^2}{2}\right]_0^{\sqrt{3}/4} \\
&= \frac{\rho\, g}{\sqrt{3}} \left(\frac{3}{8} - \frac{\sqrt{3}}{16}\right) + \rho\, g \left(\frac{3}{32}\right) = \frac{\rho\, g}{32\sqrt{3}}(12 + \sqrt{3}) = 9800 \left(\frac{\sqrt{3}}{8} + \frac{1}{32}\right) \approx 2428
\end{aligned}$$

7. [avg: 7.2/10] Consider the curve with equation $y = \dfrac{e^x + e^{-x}}{2}$ for $-\ln 2 \le x \le \ln 2$.

(a) [5 marks] Find the length of the curve.

Solution: $\dfrac{dy}{dx} = \dfrac{1}{2}\left(e^x - e^{-x}\right) \Rightarrow$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(e^x - e^{-x}\right)^2 = 1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4} = \frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2.$$

Thus the length of the curve is

$$\int_{-\ln 2}^{\ln 2} \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx = \int_{-\ln 2}^{\ln 2} \frac{e^x + e^{-x}}{2}\, dx = \int_0^{\ln 2} (e^x + e^{-x})\, dx = \left[e^x - e^{-x}\right]_0^{\ln 2} = \frac{3}{2}$$

(b) [5 marks] Find the area of the surface generated by revolving the curve about the x-axis.

Solution: the surface area is given by $\displaystyle\int_{-\ln 2}^{\ln 2} 2\pi y \sqrt{1 + (y')^2}\, dx$. Using work from part (a), we have

$$\begin{aligned}
SA &= \int_{-\ln 2}^{\ln 2} 2\pi \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) dx \\
&= \frac{\pi}{2} \int_{-\ln 2}^{\ln 2} (e^{2x} + 2 + e^{-2x})\, dx \\
&= \pi \int_0^{\ln 2} (e^{2x} + 2 + e^{-2x})\, dx \\
&= \pi \left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_0^{\ln 2} \\
&= \pi \left(2 + 2\ln 2 - \frac{1}{8}\right) \\
&= \pi \left(\frac{15}{8} + 2\ln 2\right) \approx 10.246
\end{aligned}$$

8.[avg: 7.0/10] Find the following:

(a) [5 marks] the equations of *two* continuous and differentiable functions defined on the interval $[2,6]$ with graphs that pass through the point $(3,5)$ and have arc length given by $\int_2^6 \sqrt{1+9\,x^{-4}}\,dx$.

Solution: from the arc length formula we know

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{x^4} \Rightarrow \frac{dy}{dx} = \pm\frac{3}{x^2}.$$

Function 1:

$$\begin{aligned} \frac{dy}{dx} = \frac{3}{x^2} \quad &\Rightarrow \quad y = \int \frac{3}{x^2}\,dx \\ &\Rightarrow \quad y = -\frac{3}{x} + C; \\ (x,y) = (3,5) \quad &\Rightarrow \quad 5 = -1 + C \\ &\Rightarrow \quad C = 6 \\ &\Rightarrow \quad y = -\frac{3}{x} + 6 \end{aligned}$$

Function 2:

$$\begin{aligned} \frac{dy}{dx} = -\frac{3}{x^2} \quad &\Rightarrow \quad y = -\int \frac{3}{x^2}\,dx \\ &\Rightarrow \quad y = \frac{3}{x} + D; \\ (x,y) = (3,5) \quad &\Rightarrow \quad 5 = 1 + D \\ &\Rightarrow \quad D = 4 \\ &\Rightarrow \quad y = \frac{3}{x} + 4 \end{aligned}$$

(b) [5 marks] the critical point(s) of $C(x) = \int_0^x \cos(t^2)\,dt$ on the interval $[0, \sqrt{2\pi}]$, and determine if C has a maximum or minimum value at each critical point.

Solution: use the Fundamental Theorem of Calculus to find $C'(x)$, and then proceed as usual.

$$C'(x) = \cos(x^2), \quad \text{so} \quad C''(x) = -2x\,\sin(x^2).$$

Critical Points: x must be in $[0, \sqrt{2\pi}]$.

$$\begin{aligned} C'(x) = 0 \quad &\Rightarrow \quad \cos(x^2) = 0 \\ &\Rightarrow \quad x^2 = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\ &\Rightarrow \quad x = \sqrt{\frac{\pi}{2}} \text{ or } \sqrt{\frac{3\pi}{2}} \end{aligned}$$

Now use the Second Derivative Test:

$$C''\left(\sqrt{\frac{\pi}{2}}\right) = -\sqrt{2\pi}\sin\left(\frac{\pi}{2}\right) = -\sqrt{2\pi} < 0; \; C''\left(\sqrt{\frac{3\pi}{2}}\right) = -\sqrt{6\pi}\sin\left(\frac{3\pi}{2}\right) = \sqrt{6\pi} > 0.$$

Conclusion: C has a maximum value at $x = \sqrt{\frac{\pi}{2}}$; and C has a minimum value at $x = \sqrt{\frac{3\pi}{2}}$.

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