University of Toronto

SOLUTIONS to MAT 186H1F TERM TEST 2

of **Tuesday, November 4, 2008**

Duration: 90 minutes

TOTAL MARKS: 50

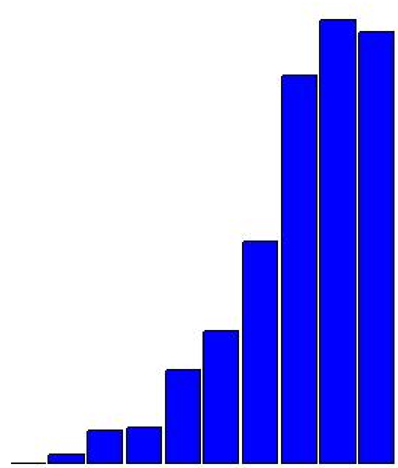
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

- Many students are still throwing away marks by using inappropriate notation or by not identifying what they are calculating.
- For Question 1: if you want to simplify $\ln x^2$ you must use $\ln x^2 = 2\ln|x|$. But that actually makes things trickier since
$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$
has to be calculated in terms of two cases: $x < 0$ or $x > 0$.
- Note in 1(b): f is *not* increasing on $(-1, 1)$; you must have two separate intervals: $(-1, 0)$ and $(0, 1)$.
- In Question 1 if you assume (for some reason) that $x > 0$ then you will forfeit half the marks in parts (b), (c), (e), (f) and (g), and will get at most 3 out of 4 for part (h). Parts (a) and (d) could still be done correctly, depending on what you've done.
- Questions 2, 3, 4 and 5 were almost carbon copies of questions from last year's test; there should have been no problems with these – for those who studied.

Breakdown of Results: 443 students wrote this test. The marks ranged from 11.7% to 100%, and the average was 75.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	24.2%
A	49.0%	80-89%	24.8 %
B	21.7%	70-79%	21.7 %
C	12.4%	60-69%	12.4 %
D	7.4%	50-59%	7.4 %
F	9.5%	40-49%	5.2 %
		30-39%	2.0%
		20-29%	1.8 %
		10-19%	0.5 %
		0-9%	0.0 %



1. [28 marks] This question has eight parts and covers three pages. Let $f(x) = \dfrac{2 + \ln x^2}{x}$.

(a) [3 marks] Verify that $f'(x) = -\dfrac{\ln x^2}{x^2}$.

Solution: use the quotient rule.

$$f'(x) = \frac{\left(\frac{2x}{x^2}\right)x - (1)(2 + \ln x^2)}{x^2} = \frac{2 - 2 - \ln x^2}{x^2} = -\frac{\ln x^2}{x^2}$$

(b) [4 marks] Find the open interval(s) on which f is increasing, and the open interval(s) on which f is decreasing.

Solution:

$$\begin{aligned} f \text{ is increasing if } f'(x) > 0 &\Leftrightarrow \ln x^2 < 0 \\ &\Leftrightarrow 0 < x^2 < 1 \\ &\Leftrightarrow 0 < |x| < 1 \\ &\Leftrightarrow -1 < x < 0 \text{ or } 0 < x < 1 \end{aligned}$$

$$\begin{aligned} f \text{ is decreasing if } f'(x) < 0 &\Leftrightarrow \ln x^2 > 0 \\ &\Leftrightarrow x^2 > 1 \\ &\Leftrightarrow |x| > 1 \\ &\Leftrightarrow x < -1 \text{ or } x > 1 \end{aligned}$$

(c) [4 marks] Find all the critical points of f and determine if they are maximum or minimum points.

Solution: $f'(x) = 0 \Leftrightarrow \ln x^2 = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$.

Using the first derivative test:

1. Since f is decreasing if $x < -1$ and f is increasing if $x > -1$, the point $(-1, f(-1)) = (-1, -2)$ is a minimum point.
2. Since f is increasing if $x < 1$ and f is decreasing if $x > 1$, the point $(1, f(1)) = (1, 2)$ is a maximum point.

Using the second derivative test, and $f''(x)$ from the next page:

1. Since $f''(-1) = 2 > 0$, the point $(-1, f(-1)) = (-1, -2)$ is a minimum point.
2. Since $f''(1) = -2 < 0$, the point the point $(1, f(1)) = (1, 2)$ is a maximum point.

(d) [3 marks] Verify that $f''(x) = \dfrac{2(-1 + \ln x^2)}{x^3}$.

Solution: use the quotient rule and the fact that $f'(x) = -\dfrac{\ln x^2}{x^2}$:

$$f''(x) = -\frac{\left(\frac{2x}{x^2}\right)x^2 - (2x)(\ln x^2)}{x^4} = -\frac{2x - 2x\ln x^2}{x^4} = \frac{2(-1+\ln x^2)}{x^3}$$

(e) [4 marks] Find the open intervals on which f is concave up, and those on which it is concave down.

Solution: $x = 0$ is a discontinuity of f (and hence of $f''(x)$) and

$$f''(x) = 0 \Leftrightarrow \ln x^2 = 1 \Leftrightarrow x^2 = e \Leftrightarrow x = \pm\sqrt{e}.$$

Check the sign of $f''(x)$ on the four intervals determined by $x = 0, x = \pm\sqrt{e}$ by using test points in each interval:

	$-\sqrt{e}$		0		$\sqrt{e}$	
$f''(-e) = -2/e^3$		$f''(-1) = 2$		$f''(1) = -2$		$f''(e) = 2/e^3$

So f is concave down if $f''(x) < 0 \Leftrightarrow x < -\sqrt{e}$ or if $0 < x < \sqrt{e}$;
and f is concave up if $f''(x) > 0 \Leftrightarrow -\sqrt{e} < x < 0$ or $x > \sqrt{e}$.

OR, solve the inequalities directly:

$$\begin{aligned} f \text{ is concave up if } f''(x) > 0 \quad &\Leftrightarrow \quad \frac{2(-1+\ln x^2)}{x^3} > 0 \\ &\Leftrightarrow \quad \ln x^2 > 1 \text{ and } x > 0 \text{ or } \ln x^2 < 1 \text{ and } x < 0 \\ &\Leftrightarrow \quad x > \sqrt{e} \text{ or } -\sqrt{e} < x < 0 \end{aligned}$$

$$\begin{aligned} f \text{ is concave down if } f''(x) < 0 \quad &\Leftrightarrow \quad \frac{2(-1+\ln x^2)}{x^3} < 0 \\ &\Leftrightarrow \quad \ln x^2 > 1 \text{ and } x < 0 \text{ or } \ln x^2 < 1 \text{ and } x > 0 \\ &\Leftrightarrow \quad x < -\sqrt{e} \text{ or } 0 < x < \sqrt{e} \end{aligned}$$

(f) [2 marks] Find all the inflection points of f, if any.

Solution:

From part (e), there are inflection points at

$$(-\sqrt{e}, f(-\sqrt{e})) = \left(-\sqrt{e}, -\frac{3}{\sqrt{e}}\right) \text{ and } (\sqrt{e}, f(\sqrt{e})) = \left(\sqrt{e}, \frac{3}{\sqrt{e}}\right)$$

(g) [4 marks] Find all the horizontal or vertical asymptotes to the graph of f, if any. Justify your answers.

Solution: $x = 0$ is a vertical asymptote since: $\lim_{x\to 0} \ln x^2 = -\infty$ and

$$\lim_{x\to 0^+} \frac{2+\ln x^2}{x} = -\infty; \quad \lim_{x\to 0^-} \frac{2+\ln x^2}{x} = \infty.$$

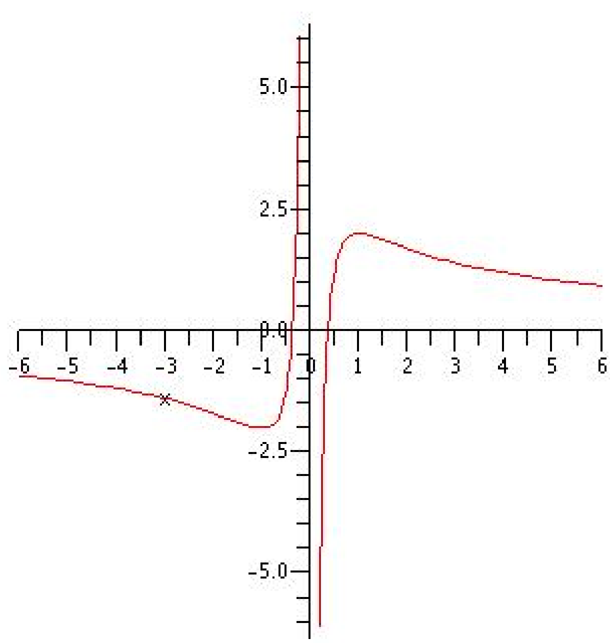
$y = 0$ is a horizontal asymptote on both sides of the graph since

$$\lim_{x\to\pm\infty} \frac{2+\ln x^2}{x} = \lim_{x\to\pm\infty} \frac{2x}{x^2} = \lim_{x\to\pm\infty} \frac{2}{x} = 0,$$

using L'Hopital's rule.

(h) [4 marks] Sketch the graph of f labelling all critical points, inflection points and asymptotes, if any.

Solution: Note you could have saved work for this entire problem by pointing out at the beginning that the graph of $f(x)$ is symmetric in the origin, since $f(-x) = -f(x)$. Then you would only have to analyze the graph for $x > 0$.



The followiing details should be included on your graph:

1. $x = 0$ is a vertical asymptote
2. $y = 0$ is a horizontal asymptote
3. $(1, 2)$ is a maximum point
4. $(-1, -2)$ is a minimum point
5. $\pm(\sqrt{e}, 3/\sqrt{e})$ are inflection points

2. [8 marks] Find the following derivatives:

(a) [3 marks] $\dfrac{d}{dx} \sin \sqrt{x+1}$

Solution: Use the chain rule.

$$\frac{d}{dx} \sin \sqrt{x+1} = (\cos \sqrt{x+1}) \frac{1}{2\sqrt{x+1}} = \frac{\cos \sqrt{x+1}}{2\sqrt{x+1}}$$

(b) [5 marks] $\dfrac{d}{dx}(x^{\sec x})$. (Assume $x > 0$.)

Solution: Let $y = x^{\sec x}$ and use logarithmic differentiation.

$$\begin{aligned} \ln y = \sec x \ln x \quad &\Rightarrow \quad \frac{y'}{y} = \sec x \tan x \ln x + \frac{\sec x}{x} \\ &\Rightarrow \quad y' = y \left(\sec x \tan x \ln x + \frac{\sec x}{x} \right) \\ &\Rightarrow \quad y' = x^{\sec x} \sec x \left(\tan x \ln x + \frac{1}{x} \right) \end{aligned}$$

3. [8 marks] Find the following limits.

(a) [3 marks] $\lim_{x \to 0} \frac{\sin(4x)}{\ln(1-x)}$

Solution: Limit is in the $\frac{0}{0}$ form. Use L'Hopital's rule.

$$\begin{aligned} \lim_{x \to 0} \frac{\sin(4x)}{\ln(1-x)} &= \lim_{x \to 0} \frac{4\cos(4x)}{\frac{-1}{1-x}} \\ &= -4 \end{aligned}$$

(b) [5 marks] $\lim_{x \to \infty} (2x + 3e^x)^{5/x}$

Solution: Limit is in the ∞^0 form. Let the limit be L.

$$\begin{aligned} \ln L &= \lim_{x \to \infty} \frac{5}{x} \ln(2x + 3e^x) \\ &= 5 \lim_{x \to \infty} \frac{\ln(2x + 3e^x)}{x}, \quad \text{which is in } \frac{\infty}{\infty} \text{ form} \\ &= 5 \lim_{x \to \infty} \frac{2 + 3e^x}{2x + 3e^x} \quad (\text{ by L'Hopital's rule}) \\ &= 5 \lim_{x \to \infty} \frac{3e^x}{2 + 3e^x} \quad (\text{by L'Hopital's rule again}) \\ &= 5 \lim_{x \to \infty} \frac{3e^x}{3e^x} \quad (\text{by L'Hopital's rule yet again}) \\ &= 5 \\ \Rightarrow L &= e^5 \end{aligned}$$

4. [8 marks] Find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(x, y) = (1, \pi)$ if

$$\sin(xy) = x + \cos y.$$

Solution: Differentiate implicitly.

$$\begin{aligned} \sin(xy) &= x + \cos y & (1) \\ \Rightarrow \cos(xy)(y + xy') &= 1 - y' \sin y & (2) \\ \Rightarrow -\sin(xy)(y + xy')^2 + \cos(xy)(y' + y' + xy'') &= -(y')^2 \cos y - y'' \sin y & (3) \end{aligned}$$

To find y' at $(x, y) = (1, \pi)$, substitute $x = 1$ and $y = \pi$ into equation (2) and solve for y' :

$$\begin{aligned} \cos(\pi)(\pi + y') = 1 - y' \sin \pi \quad &\Rightarrow \quad \pi + y' = -1 \\ &\Rightarrow \quad y' = -1 - \pi \end{aligned}$$

To find y'' at $(x, y) = (1, \pi)$, substitute $x = 1, y = \pi$ and $y' = -1-\pi$ into equation (3) and solve for y'' :

$$\begin{aligned} -\sin(\pi)(\pi + (-1 - \pi))^2 + \cos(\pi)(2(-1 - \pi) + y'') &= -(-1 - \pi)^2 \cos \pi - y'' \sin \pi \\ \Rightarrow -(-2 - 2\pi + y'') &= (-1 - \pi)^2 \\ \Rightarrow 2(1 + \pi) - y'' &= (1 + \pi)^2 \\ \Rightarrow -y'' &= (1 + \pi)^2 - 2(1 + \pi) \\ \Rightarrow -y'' &= 1 + 2\pi + \pi^2 - 2 - 2\pi \\ \Rightarrow -y'' &= \pi^2 - 1 \\ \Rightarrow y'' &= 1 - \pi^2 \end{aligned}$$

Actually, its neater to substitute $x = 1, y = \pi$ into equation (3) and solve for y'' in terms of y' :

$$\begin{aligned} -\sin(\pi)(\pi + y')^2 + \cos(\pi)(2y' + y'') &= -(y')^2 \cos \pi - y'' \sin \pi \\ \Rightarrow -(2y' + y'') &= (y')^2 \\ \Rightarrow -y'' &= (y')^2 + 2y' \\ \Rightarrow -y'' &= y'(y' + 2) \\ \Rightarrow y'' &= -y'(y' + 2) \\ \text{Now substitute } y' = -1 - \pi\text{:} \quad y'' &= (1 + \pi)(1 - \pi) = 1 - \pi^2 \end{aligned}$$

5. [8 marks]

(a) [4 marks] Find an approximation to $7^{\frac{2}{3}}$ by using the linear approximation of $f(x) = x^{\frac{2}{3}}$ at $a = 8$. (Express your answer to five decimal places.)

Solution: $f'(x) = \dfrac{2}{3}\dfrac{1}{x^{1/3}}$, so the equation of the tangent line to f at $a = 8$ is

$$\begin{aligned} \frac{y - f(8)}{x - 8} = f'(8) \quad &\Leftrightarrow \quad \frac{y-4}{x-8} = \frac{2}{3}\frac{1}{8^{1/3}} \\ &\Leftrightarrow \quad y = 4 + \frac{1}{3}(x-8). \\ \text{So} \quad 7^{2/3} = f(7) \quad &\simeq \quad 4 + \frac{1}{3}(7-8) \\ &= \quad \frac{11}{3} \simeq 3.66667 \end{aligned}$$

(b) [4 marks] Find an approximation to $7^{\frac{2}{3}}$ by applying Newton's method to the equation

$$x^{3/2} - 7 = 0;$$

start with $x_0 = 4$ and compute x_1 and x_2. (Express your answers to five decimal places.)

Solution: $f(x) = x^{3/2} - 7$; $f'(x) = \dfrac{3}{2}\sqrt{x}$. So the recursive formula for Newton's method is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^{3/2} - 7}{\frac{3}{2}\sqrt{x_n}} \\ &= \frac{x_n^{3/2} + 14}{3\sqrt{x_n}} \end{aligned}$$

$$x_0 = 4 \quad \Rightarrow \quad x_1 = \frac{22}{6} \simeq 3.66667$$

and

$$x_1 = \frac{11}{3} \text{ or } 3.66667 \quad \Rightarrow \quad x_2 \simeq 3.65931$$