University of Toronto

SOLUTIONS to MAT186H1F TERM TEST 1

of **Tuesday, October 6, 2009**

General Comments about the Test:

- Many students seem to have missed the instructions on the front page of the test:

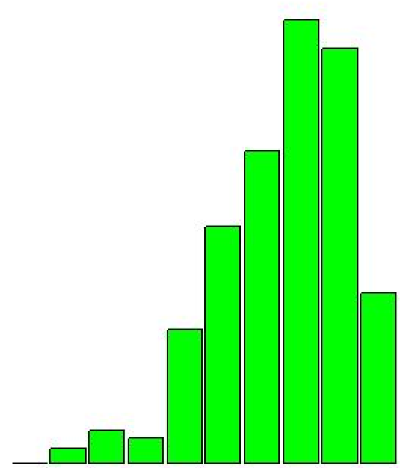
  Present your solutions in the space provided. This means: show your work!

  You won't get full marks on a written test question unless you explain your solution clearly and completely.

- Questions 1, 2(a), 4, 5(a), 6 and 7 are all completely routine.
- In Question 2(a), $0 < \cos^{-1}(-2/3) < \pi$, so the sine of this angle must be positive. A lot of students gave two answers, a negative and a positive one.
- Questions 2(b) and 5(b) were the hardest questions on the paper. 5(b) starts out routinely enough–rationalize–but then you have to realize that $|x| = \sqrt{x^2} = -x$, since $x \to -\infty$, to finish it correctly. 2(b) is a variation on a WileyPlus assigned question about inverses.
- The limits in Questions 3(a) and 3(b) are not the same; a lot of students ignored the absolute value in both parts and got the same answers for both parts–which completely misses the point of this question–and is a major error.
- In Questions 6(b) and 7(a) you have to clearly explain how the Intermediate Value Theorem and the Squeezing Theorem, respectively, apply, to get full marks.
- In Question 7(b) you have to clearly show how the basic trig limit can be invoked to finish the calculations.

Breakdown of Results: 491 students wrote this test. The marks ranged from 15% to 100%, and the average was 70%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	9.6%
A	32.8%	80-89%	23.2%
B	24.9%	70-79%	24.9%
C	17.5%	60-69%	17.5%
D	13.2%	50-59%	13.2%
F	11.6%	40-49%	7.5%
		30-39%	1.4%
		20-29%	1.8%
		10-19%	0.8%
		0-9%	0.0%



1. [9 marks] Suppose $\cos x = \dfrac{1}{5}$ and $\sin x < 0$. Find the exact values of the following:

(a) [2 marks] $\sin x$

Solution:

$$\begin{aligned}
\sin x &= -\sqrt{1-\cos^2 x} \\
&= -\sqrt{1-\left(\frac{1}{5}\right)^2} \\
&= -\sqrt{1-\left(\frac{1}{25}\right)} \\
&= -\sqrt{\frac{24}{25}} \\
&= -\frac{2}{5}\sqrt{6}
\end{aligned}$$

(b) [3 marks] $\sin(2x)$

Solution:

$$\begin{aligned}
\sin(2x) &= 2\sin x\cos x \\
&= 2\left(-\frac{2}{5}\sqrt{6}\right)\left(\frac{1}{5}\right) \\
&= -\frac{4}{25}\sqrt{6}
\end{aligned}$$

(c) [4 marks] $\cos\left(x+\dfrac{\pi}{6}\right)$

Solution:

$$\begin{aligned}
\cos\left(x+\frac{\pi}{6}\right) &= \cos x\,\cos\frac{\pi}{6} - \sin x\,\sin\frac{\pi}{6} \\
&= \left(\frac{1}{5}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(-\frac{2}{5}\sqrt{6}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{3}}{10} + \frac{\sqrt{6}}{5}
\end{aligned}$$

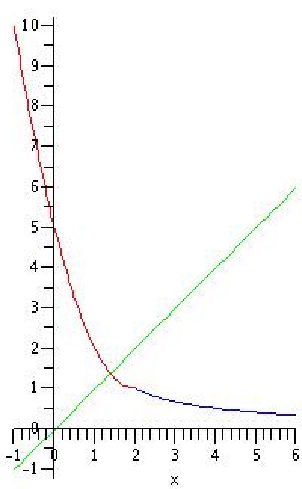
2(a) [4 marks] Find the exact value of $\sin\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$.

Solution: let $\theta = \cos^{-1}\left(-\frac{2}{3}\right)$. Then $\cos\theta = -\frac{2}{3}$ and $\frac{\pi}{2} < \theta < \pi$. So

$$\begin{aligned}\sin\left(\cos^{-1}\left(-\frac{2}{3}\right)\right) &= \sin\theta \\ &= \sqrt{1-\cos^2\theta} \\ &= \sqrt{1-\left(-\frac{2}{3}\right)^2} \\ &= \sqrt{1-\frac{4}{9}} \\ &= \frac{\sqrt{5}}{3}\end{aligned}$$

2.(b) [5 marks] Find the formula for $f^{-1}(x)$ if $f(x) = \begin{cases} (x-2)^2+1, & \text{if } x < 2 \\ \frac{2}{x}, & \text{if } x \geq 2 \end{cases}$.

Solution: the graph of f is shown below, along with the line $y = x$. Then



$$x = \frac{2}{y} \Leftrightarrow y = \frac{2}{x}$$

and

$$x = (y-2)^2 + 1 \Leftrightarrow y = 2 \pm \sqrt{x-1}.$$

Since $(0,5)$ is on the graph of f, $(5,0)$ must be on the graph of f^{-1}. So

$$f^{-1}(x) = \begin{cases} -\sqrt{x-1}+2, & \text{if } x > 1 \\ \frac{2}{x}, & \text{if } 0 < x \leq 1 \end{cases}.$$

Aside: other observations that can help you organize this question: the range of f is $y > 0$, so the domain of f^{-1} must be $x > 0$; and the domain of f is $\mathbb{R}$, so the range of f^{-1} must be $\mathbb{R}$. The point $(2,1)$ is the dividing point for the two cases of f, so the point $(1,2)$ is the dividing point for the two cases of f^{-1}.

3. [9 marks] Find the following limits.

(a) [3 marks] $\lim_{x\to 2^+} \frac{|x^2-x-2|}{x-2}$

Solution: factor and simplify.

$$\begin{aligned}
\lim_{x\to 2^+} \frac{|x^2-x-2|}{x-2} &= \lim_{x\to 2^+} \frac{|(x-2)(x+1)|}{x-2} = \lim_{x\to 2^+} \frac{|x-2||x+1|}{x-2} \\
&= \lim_{x\to 2^+} \frac{(x-2)(x+1)}{x-2}, \text{ since } x > 2 \\
&= \lim_{x\to 2^+} (x+1) = 3
\end{aligned}$$

(b) [3 marks] $\lim_{x\to 2^-} \frac{|x^2-x-2|}{x-2}$

Solution: factor and simplify.

$$\begin{aligned}
\lim_{x\to 2^-} \frac{|x^2-x-2|}{x-2} &= \lim_{x\to 2^-} \frac{|(x-2)(x+1)|}{x-2} = \lim_{x\to 2^-} \frac{|x-2||x+1|}{x-2} \\
&= \lim_{x\to 2^-} \frac{-(x-2)(x+1)}{x-2}, \text{ since } x < 2 \\
&= \lim_{x\to 2^-} -(x+1) = -3
\end{aligned}$$

(c) [3 marks] $\lim_{x\to 0^-} \frac{\sin x}{1-\cos x}$

Solution: multiply through by $1+\cos x$.

$$\begin{aligned}
\lim_{x\to 0^-} \frac{\sin x}{1-\cos x} &= \lim_{x\to 0^-} \frac{\sin x\,(1+\cos x)}{(1-\cos x)(1+\cos x)} \\
&= \lim_{x\to 0^-} \frac{\sin x\,(1+\cos x)}{1-\cos^2 x} \\
&= \lim_{x\to 0^-} \frac{\sin x\,(1+\cos x)}{\sin^2 x} \\
&= \lim_{x\to 0^-} \frac{1+\cos x}{\sin x}, \text{ which is in } \frac{2}{0^-} \text{ form} \\
&= -\infty
\end{aligned}$$

4. [8 marks] Find the following limits.

(a) [3 marks] $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

Solution: factor and simplify.

$$\begin{aligned}
\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} \\
&= \lim_{x\to 9} \sqrt{x}+3 \\
&= 6
\end{aligned}$$

Alternate Solution: rationalize and simplify.

(b) [5 marks] $\lim_{x\to 9} \frac{\sqrt{3x+9}-6}{\sqrt{x}-3}$

Solution: rationalize twice and simplify.

$$\begin{aligned}
\lim_{x\to 9} \frac{\sqrt{3x+9}-6}{\sqrt{x}-3} &= \lim_{x\to 9} \frac{(\sqrt{3x+9}-6)(\sqrt{3x+9}+6)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{3x+9}+6)(\sqrt{x}+3)} \\
&= \lim_{x\to 9} \frac{(3x+9-36)(\sqrt{x}+3)}{(x-9)(\sqrt{3x+9}+6)} \\
&= \lim_{x\to 9} \frac{(3x-27)(\sqrt{x}+3)}{(x-9)(\sqrt{3x+9}+6)} \\
&= \lim_{x\to 9} \frac{3(x-9)(\sqrt{x}+3)}{(x-9)(\sqrt{3x+9}+6)} \\
&= 3\lim_{x\to 9} \frac{\sqrt{x}+3}{\sqrt{3x+9}+6} \\
&= 3\left(\frac{6}{6+6}\right) \\
&= \frac{3}{2}
\end{aligned}$$

Alternate Solution:

$$\lim_{x\to 9} \frac{\sqrt{3x+9}-6}{\sqrt{x}-3} = \lim_{x\to 9} \frac{\sqrt{3x+9}-6}{x-9} \cdot \frac{x-9}{\sqrt{x}-3} = 6\lim_{x\to 9} \frac{\sqrt{3x+9}-6}{x-9},$$

by part (a). To finish, rationalize and simplify in the remaining limit.

5. [8 marks] Find the following limits:

(a) [3 marks] $\lim_{x\to\infty} \dfrac{x^2+3x-14}{4-3x^2}$

Solution: divide through by the highest power of x.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+3x-14}{4-3x^2} &= \lim_{x\to\infty} \frac{1+3/x-14/x^2}{4/x^2-3} \\
&= \frac{1+0-0}{0-3} \\
&= -\frac{1}{3}
\end{aligned}$$

(b) [5 marks] $\lim_{x\to-\infty} \sqrt{x^2+9x}+x$

Solution: rationalize and simplify.

$$\begin{aligned}
\lim_{x\to-\infty} \sqrt{x^2+9x}+x &= \lim_{x\to-\infty} \frac{(\sqrt{x^2+9x}+x)(\sqrt{x^2+9x}-x)}{\sqrt{x^2+9x}-x} \\
&= \lim_{x\to-\infty} \frac{x^2+9x-x^2}{\sqrt{x^2+9x}-x} \\
&= \lim_{x\to-\infty} \frac{9x}{\sqrt{x^2+9x}-x} \\
&= \lim_{x\to-\infty} \frac{9}{\frac{\sqrt{x^2+9x}}{x}-1} \\
&= \lim_{x\to-\infty} \frac{9}{-\sqrt{\frac{x^2+9x}{x^2}}-1}, \text{ since } x<0 \\
&= \lim_{x\to-\infty} \frac{9}{-\sqrt{1+9/x}-1} \\
&= \frac{9}{-\sqrt{1+0}-1} \\
&= -\frac{9}{2}
\end{aligned}$$

6.(a) [5 marks] Find all the discontinuities of $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 1}$ and determine if each discontinuity is removable, jump or infinite.

Solution: discontinuities: $x^2 - 1 = 0 \Leftrightarrow x = \pm 1$.

At $x = 1$:

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^2 - 6x + 5}{x^2 - 1} \\ &= \lim_{x\to 1} \frac{(x-5)(x-1)}{(x+1)(x-1)} \\ &= \lim_{x\to 1} \frac{x-5}{x+1} \\ &= -2,\end{aligned}$$

so f has a removable discontinuity at $x = 1$.

At $x = -1$:

$$\begin{aligned}\lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} \frac{x^2 - 6x + 5}{x^2 - 1} \\ &= \lim_{x\to -1^+} \frac{(x-5)(x-1)}{(x+1)(x-1)} \\ &= \lim_{x\to -1^+} \frac{x-5}{x+1}, \text{ which is in } \frac{-6}{0^+} \text{ form} \\ &= -\infty,\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} \frac{x^2 - 6x + 5}{x^2 - 1} \\ &= \lim_{x\to -1^-} \frac{(x-5)(x-1)}{(x+1)(x-1)} \\ &= \lim_{x\to -1^-} \frac{x-5}{x+1}, \text{ which is in } \frac{-6}{0^-} \text{ form} \\ &= \infty,\end{aligned}$$

so f has an infinite discontinuity at $x = -1$.

6.(b) [4 marks] Using the Intermediate Value Theorem, explain succinctly why the equation $x - \cos x = 0$ has at least one solution in the interval $[0, \pi/2]$.

Solution: let $f(x) = x - \cos x$, which is a continuous function for all x. Observe that

$$f(0) = 0 - \cos 0 = 0 - 1 = -1 < 0$$

and

$$f(\pi/2) = \pi/2 - \cos \pi/2 = \pi/2 - 0 = \pi/2 > 0.$$

So by the Intermediate Value Theorem, there is a number $c \in (0, \pi/2)$ such that

$$f(c) = 0 \Leftrightarrow c - \cos c = 0.$$

7. [8 marks]

(a) [4 marks] Use the Squeezing Theorem to find $\lim_{x\to 2} (x-2)^2 \sin(\sec(\pi/x))$.

Solution: $\sec(\pi/2)$ is not defined. Use the Squeezing Theorem.

$$\begin{aligned} & x \neq 2 \Rightarrow -1 \leq \sin\left(\sec(\pi/x)\right) \leq 1 \\ \Rightarrow\quad & -(x-2)^2 \leq (x-2)^2 \sin\left(\sec(\pi/x)\right) \leq (x-2)^2 \end{aligned}$$

Both

$$\lim_{x\to 2} -(x-2)^2 = 0 \text{ and } \lim_{x\to 2}(x-2)^2 = 0,$$

so by the Squeezing Theorem,

$$\lim_{x\to 2} (x-2)^2 \sin(\sec(\pi/x)) = 0$$

as well.

(b) [4 marks] Find $\lim_{x\to 0} \dfrac{\sin(-4x)}{\sin(7x)}$.

Solution: make use of the basic trig limit $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$.

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(-4x)}{\sin(7x)} &= \lim_{x\to 0} \left(-\frac{4}{7} \cdot \frac{\sin(-4x)}{(-4x)} \cdot \frac{(7x)}{\sin(7x)} \right) \\ &= -\frac{4}{7} \cdot \lim_{x\to 0} \frac{\sin(-4x)}{(-4x)} \cdot \lim_{x\to 0} \frac{(7x)}{\sin(7x)} \\ &= -\frac{4}{7} \cdot 1 \cdot \frac{1}{1}, \text{ making use of the basic trig limit} \\ &= -\frac{4}{7} \end{aligned}$$