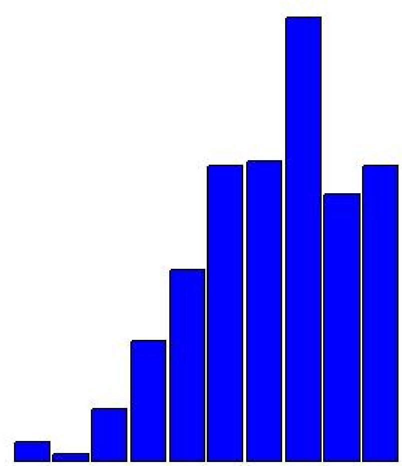
University of Toronto
SOLUTIONS to MAT186H1F TERM TEST 1
of **Tuesday, October 19, 2010**

General Comments about the Test:

- In a written test you must explain what you are doing to get full credit. The answer by itself is worth very little if you don't explain how you got it. You can't just plop down formulas and expect the marker to figure out what you are doing. You are supposed to make it clear what you are doing.
- Some students were confused: $\sin^{-1} x$, $\sec^{-1} x$ and $\tan^{-1} x$ are inverse trigonometric functions, NOT reciprocals!
- In Question 6(a) you have to indicate how continuity is used to get the equations for m and b, to get full marks.
- In Question 6(b) you don't have to use the definition of derivative at $x = 2$ or $x = -1$, although we would have gladly accepted that. All you have to check is for which values of x the derivative f' is continuous.
- In Question 7(a) you have to state that the polynomial is continuous on the given interval; otherwise the Intermediate Value Theorem does not apply.

Breakdown of Results: 499 students wrote this test. The marks ranged from 5% to 100%, and the average was 66.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	14.8 %
A	28.2 %	80-89%	13.4 %
B	22.2 %	70-79%	22.2 %
C	15.0 %	60-69%	15.0 %
D	14,8 %	50-59%	14.8 %
F	19.6 %	40-49%	9.6 %
		30-39%	6.0 %
		20-29%	2.6 %
		10-19%	0.4 %
		0-9%	1.0 %



1. [8 marks] Find $\dfrac{dy}{dx}$ if

   (a) [3 marks] $y = x \tan x$.

   Solution: use the product rule.

$$\frac{dy}{dx} = \tan x + x \sec^2 x$$

   (b) [5 marks] $y = x^{\sin x}$, if $x > 0$.

   Solution: use logarithmic differentiation.

$$\begin{aligned} y = x^{\sin x} \quad &\Rightarrow \quad \ln y = \sin x \ln x \\ &\Rightarrow \quad \frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x} \\ &\Rightarrow \quad y' = x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right) \end{aligned}$$

2. [8 marks] Find all the points on the graph with equation $y = 4 - x^2$ such that the tangent line to the graph at each of these points passes through the point $(2, 9)$.

Solution: let the point of contact be $(a, 4 - a^2)$. Let $f(x) = 4 - x^2$. Then the slope of the tangent line to the parabola from the point $(2, 9)$ is

$$\frac{9 - (4 - a^2)}{2 - a}$$

using the slope between two points, and is also

$$f'(a) = -2a$$

using calculus. Hence

$$\frac{9 - (4 - a^2)}{2 - a} = -2a \Leftrightarrow 5 + a^2 = -4a + 2a^2 \Leftrightarrow a^2 - 4a - 5 = 0 \Leftrightarrow a = 5 \text{ or } -1.$$

So the two points on the parabola from which the tangent line passes through the point $(2, 9)$ are

$$(-1, 3) \text{ and } (5, -21).$$

3. [8 marks] Find the value of the following derivatives at the indicated point.

(a) [4 marks] $\dfrac{d\sec^{-1}(e^{x/2})}{dx}$; at $x = \ln 5$.

Solution: use the chain rule.

$$\begin{aligned}\frac{d\sec^{-1}(e^{x/2})}{dx} &= \frac{1}{e^{x/2}\sqrt{(e^{x/2})^2-1}}\left(\frac{e^{x/2}}{2}\right)\\ &= \frac{1}{2\sqrt{e^x-1}}\end{aligned}$$

At $x = \ln 5$ this simplifies to

$$\frac{dy}{dx} = \frac{1}{2\sqrt{5-1}} = \frac{1}{4}.$$

(b) [4 marks] $\dfrac{d\sin^{-1}(1/x)}{dx}$; at $x = -\sqrt{2}$.

Solution: use chain rule.

$$\begin{aligned}\frac{d\sin^{-1}(1/x)}{dx} &= \frac{1}{\sqrt{1-(1/x)^2}}\left(-\frac{1}{x^2}\right)\\ &= -\frac{|x|}{\sqrt{x^2-1}}\frac{1}{x^2}\end{aligned}$$

At $x = -\sqrt{2}$ this simplifies to

$$\frac{dy}{dx} = -\frac{\sqrt{2}}{\sqrt{2-1}}\frac{1}{2} = -\frac{1}{\sqrt{2}}.$$

4. [8 marks] Let $f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right)$, for $-1 < x < 1$. Find:

(a) [4 marks] $f^{-1}(x)$

Solution: interchange x and y and solve for y:

$$\begin{aligned} x = \tan^{-1}\left(\frac{1-y}{1+y}\right) \quad &\Rightarrow \quad \tan x = \frac{1-y}{1+y} \\ &\Rightarrow \quad \tan x + y\tan x = 1 - y \\ &\Rightarrow \quad y(\tan x + 1) = 1 - \tan x \\ &\Rightarrow \quad y = \frac{1-\tan x}{1+\tan x} \end{aligned}$$

So the formula for f^{-1} is

$$f^{-1}(x) = \frac{1-\tan x}{1+\tan x}.$$

(b) [4 marks] $(f^{-1})'(\pi/4)$

Solution: use the quotient rule.

$$(f^{-1})'(x) = \frac{-\sec^2 x\,(1+\tan x) - \sec^2 x\,(1-\tan x)}{(1+\tan x)^2}$$

Consequently,

$$(f^{-1})'(\pi/4) = \frac{-2\,(1+1) - 2\,(1-1)}{(1+1)^2} = -1.$$

Alternate Solution: use $f^{-1}(\pi/4) = 0$, since $f(0) = \tan^{-1} 1$. You can differentiate implicitly to obtain

$$\tan(f(x)) = \frac{1-x}{1+x} \Rightarrow \sec^2(f(x))f'(x) = \frac{-1-x-1+x}{(1+x)^2} = -\frac{2}{(1+x)^2}$$

Let $x = 0$:

$$\sec^2(\pi/4)f'(0) = -2 \Rightarrow f'(0) = -1,$$

and so

$$\left(f^{-1}\right)'(\pi/4) = \frac{1}{f'(f^{-1}(\pi/4))} = \frac{1}{f'(0)} = -1.$$

5. [8 marks] Find the following limits:

(a) [3 marks] $\lim_{x\to\infty} \left(\dfrac{2+3x-27x^2}{1+8x^2} \right)^{1/3}$

Solution: divide through by the highest power of x.

$$\begin{aligned}
\lim_{x\to\infty} \left(\frac{2+3x-27x^2}{1+8x^2} \right)^{1/3} &= \left(\lim_{x\to\infty} \frac{2+3x-27x^2}{1+8x^2} \right)^{1/3} \\
&= \left(\lim_{x\to\infty} \frac{2/x^2+3/x^2-27}{1/x^2+8} \right)^{1/3} \\
&= \left(\lim_{x\to\infty} \frac{0+0-27}{0+8} \right)^{1/3} \\
&= \left(-\frac{27}{8} \right)^{1/3} \\
&= -\frac{3}{2}
\end{aligned}$$

(b) [5 marks] $\lim_{x\to\infty} \left(\sqrt{x^2+10x} - x \right)$

Solution: rationalize and simplify.

$$\begin{aligned}
\lim_{x\to\infty} \left(\sqrt{x^2+10x} - x \right) &= \lim_{x\to-\infty} \frac{(\sqrt{x^2+10x}-x)(\sqrt{x^2+10x}+x)}{\sqrt{x^2+10x}+x} \\
&= \lim_{x\to-\infty} \frac{x^2+10x-x^2}{\sqrt{x^2+10x}+x} \\
&= \lim_{x\to-\infty} \frac{10x}{\sqrt{x^2+10x}+x} \\
&= \lim_{x\to-\infty} \frac{10}{\frac{\sqrt{x^2+10x}}{x}+1} \\
&= \lim_{x\to-\infty} \frac{10}{\sqrt{\frac{x^2+10x}{x^2}}+1}, \text{ since } x>0 \\
&= \lim_{x\to-\infty} \frac{10}{\sqrt{1+10/x}+1} \\
&= \frac{10}{\sqrt{1+0}+1} = 5
\end{aligned}$$

6. [9 marks] Suppose $f(x) = \begin{cases} x^2+7 & \text{if} \quad x > 2 \\ mx+b & \text{if} \quad -1 < x \leq 2 \\ 2x^3 - x & \text{if} \quad x \leq -1 \end{cases}$

(a) [5 marks] Find the values of m and b so that f is continuous everywhere.

Solution: since f is comprised of polynomial functions, which are continuous everywhere, you only need to make the one-sided limits at $x = 2$ agree, and at $x = -1$. At $x = 2$:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \Rightarrow 2m + b = 11;$$

at $x = -1$:

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x) \Rightarrow -m + b = -1.$$

Solving for m and b gives $m = 4, b = 3$.

(b) [4 marks] Assuming that f is continuous everywhere, find all points, if any, at which f is not differentiable.

Solution: with $m = 4$ and $b = 3$ we have:

$$f(x) = \begin{cases} x^2+7 & \text{if} \quad x > 2 \\ 4x+3 & \text{if} \quad -1 < x \leq 2 \\ 2x^3 - x & \text{if} \quad x \leq -1 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & \text{if} \quad x > 2 \\ 4 & \text{if} \quad -1 < x < 2 \\ 6x^2 - 1 & \text{if} \quad x < -1 \end{cases}$$

So the only two possible points at which f might not be differentiable are $x = 2$ and $x = -1$. At $x = 2$,

$$\lim_{x \to 2^+} f'(x) = 4 = \lim_{x \to 2^-} f'(x),$$

so f is differentiable at $x = 2$. But at $x = -1$,

$$\lim_{x \to -1^+} f'(x) = 4 \neq 5 = \lim_{x \to -1^-} f'(x),$$

so f is not differentiable at $x = -1$. Conclusion: f is not differentiable at $x = -1$, only.

7.(a) [4 marks] Use the Intermediate Value Theorem to explain, clearly and concisely, why the equation $x^3 + x^2 - 2x - 1 = 0$ has at least one solution in the interval $[-1, 1]$.

Solution: let $f(x) = x^3 + x^2 - 2x - 1$, which is a polynomial function and so is continuous for all x. Now consider $f(x)$ on the closed interval $[-1, 1[$:

$$f(-1) = 1 > 0 \text{ and } f(1) = -1 < 0.$$

So by the Intermediate Value Theorem there is a number $c \in (-1, 1)$ such that

$$f(c) = 0 \Leftrightarrow x^3 + c^2 - 2c - 1 = 0.$$

7.(b) [4 marks] Find $\lim_{x\to 0} \frac{x^2}{1 - \cos(3x)}$

Solution: make use of the basic trig limit $\lim_{h\to 0} \frac{\sin h}{h} = 1$.

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2}{1-\cos(3x)} &= \lim_{x\to 0} \frac{x^2(1+\cos(3x))}{(1-\cos(3x))(1+\cos(3x))} \\
&= \lim_{x\to 0} \frac{x^2(1+\cos(3x))}{1-\cos^2(3x)} \\
&= \lim_{x\to 0} \frac{x^2(1+\cos(3x))}{\sin^2(3x)} \\
&= \lim_{x\to 0}(1+\cos(3x)) \left(\frac{1}{3} \lim_{x\to 0} \frac{3x}{\sin(3x)}\right)^2 \\
&= \frac{2}{9} \lim_{x\to 0} \frac{3x}{\sin(3x)} \\
&= \frac{2}{9} \lim_{h\to 0} \frac{h}{\sin h}, \text{ if } h = 3x \\
&= \frac{2}{9} \cdot 1 \\
&= \frac{2}{9}
\end{aligned}$$

8. [9 marks] Find both $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(x, y) = (1, 1)$ if $\quad 2\sqrt{x} - \sqrt{y} = xy.$

Solution: differentiate implicitly.

$$2\sqrt{x} - \sqrt{y} = xy \Rightarrow \frac{2}{2\sqrt{x}} - \frac{1}{2\sqrt{y}}\frac{dy}{dx} = y + x\frac{dy}{dx} \Rightarrow \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{y}}\frac{dy}{dx} = y + x\frac{dy}{dx}$$

At $(x, y) = (1, 1)$, this becomes

$$1 - \frac{1}{2}\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 0.$$

To find $\dfrac{d^2y}{dx^2}$, differentiate implicitly once more:

$$\frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{y}}\frac{dy}{dx} = y + x\frac{dy}{dx} \Rightarrow -\frac{1}{2x^{3/2}} + \frac{1}{4y^{3/2}}\left(\frac{dy}{dx}\right)^2 - \frac{1}{2\sqrt{y}}\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2}.$$

At $(x, y) = (1, 1)$ and $\dfrac{dy}{dx} = 0$, so

$$-\frac{1}{2} + \frac{1}{4}(0)^2 - \frac{1}{2}\frac{d^2y}{dx^2} = 0 + 0 + \frac{d^2y}{dx^2} \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{3}.$$

Alternate Solution: not recommended. Solve for y as a function of x and differentiate explicitly.

$$xy + \sqrt{y} - 2\sqrt{x} = 0 \Rightarrow \sqrt{y} = \frac{-1 \pm \sqrt{1 + 8x\sqrt{x}}}{2x}.$$

Since $y = 1$ when $x = 1$, take the positive square root and square both sides:

$$y = \frac{1 - 2\sqrt{1 + 8x\sqrt{x}} + 1 + 8x\sqrt{x}}{4x^2} = \frac{1 - \sqrt{1 + 8x\sqrt{x}} + 4x\sqrt{x}}{2x^2}.$$

Whence, after much work:

$$\frac{dy}{dx} = -\frac{\sqrt{1 + 8x^{3/2}} - 1 - 5x^{3/2} + x^{3/2}\sqrt{1 + 8x^{3/2}}}{x^3\sqrt{1 + 8x^{3/2}}}$$

and

$$\frac{d^2y}{dx^2} = \frac{3}{2}\left[\frac{-25x^2 - 60x^{7/2} + 17\sqrt{1 + 8x^{3/2}}x^2 + 8\sqrt{1 + 8x^{3/2}}x^{7/2} + 2\sqrt{1 + 8x^{3/2}}\sqrt{x} - 2\sqrt{x}}{x^{9/2}(1 + 8x^{3/2})^{3/2}}\right]$$

For $x = 1, y = 1$ we obtain the same answers: $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = -\dfrac{1}{3}.$