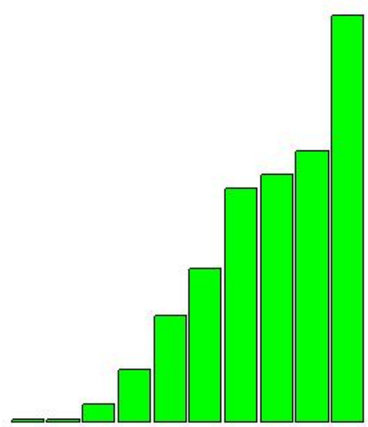
University of Toronto
Solutions to **MAT187H1S TERM TEST 1**
of **Thursday, January 31, 2013**
Duration: 100 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.
General Comments:

1. The results on this test were very good. Almost half the class had 80% or better.
2. There were 11 perfect papers.
3. But 63 students failed this test. These students are in trouble since the course will only get harder.
4. Indefinite integrals require a constant of integration; definite integrals do not.
5. If in Questions 1 or 3 you make a substitution in the definite integral, then you must change the limits of integration as you change the variable. Theorem 5.9.1 on page 391 of the textbook spells it out:
$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du,$$
where the substitution is $u = g(x)$. Doing this will actually save you work.
6. The whole point of the Question 5 is how to handle the minus sign in $\sqrt{4x - x^2}$.
Since $4x - x^2 \geq 0 \Rightarrow -2 \leq x - 2 \leq 2$, letting $x - 2 = 2\sec\theta$ or $2\tan\theta$ makes no sense at all. This question could also be done by letting $u = \sqrt{x}$, followed by the trig substitution $u = 2\sin\theta$.

Breakdown of Results: 518 students wrote this test. The marks ranged from 8.3% to 100%, and the average was 73.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	27.2%
A	45.3%	80-89%	18.1%
B	16.7%	70-79%	16.7%
C	15.6%	60-69%	15.6%
D	10.2%	50-59%	10.2%
F	12.2%	40-49%	7.1%
		30-39%	3.5%
		20-29%	1.2%
		10-19%	0.2%
		0-9%	0.2%



Formulas you may find useful. DO NOT TEAR THIS PAGE FROM THE TEST.

1. $\int e^u \, du = e^u + C$
2. $\displaystyle\int u^n \, du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
3. $\displaystyle\int \frac{1}{u} \, du = \ln|u| + C$
4. $\displaystyle\int \cos u \, du = \sin u + C$
5. $\displaystyle\int \sin u \, du = -\cos u + C$
6. $\displaystyle\int \sec^2 u \, du = \tan u + C$
7. $\displaystyle\int \sec u \tan u \, du = \sec u + C$
8. $\displaystyle\int \csc^2 u \, du = -\cot u + C$
9. $\displaystyle\int \csc u \cot u \, du = -\csc u + C$
10. $\displaystyle\int \tan u \, du = \ln|\sec u| + C$
11. $\displaystyle\int \sec u \, du = \ln|\sec u + \tan u| + C$
12. $\displaystyle\int \cot u \, du = -\ln|\csc u| + C$
13. $\displaystyle\int \csc u \, du = -\ln|\csc u + \cot u| + C$
14. $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}} \, du = \sin^{-1} \frac{u}{a} + C = \arcsin \frac{u}{a} + C$
15. $\displaystyle\int \frac{1}{a^2 + u^2} \, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C = \frac{1}{a} \arctan \frac{u}{a} + C$
16. $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}} \, du = \frac{1}{a} \sec^{-1} \left|\frac{u}{a}\right| + C = \frac{1}{a} \text{arcsec} \left|\frac{u}{a}\right| + C$
17. $\displaystyle\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$
18. $\sin^2 \theta + \cos^2 \theta = 1$
19. $\tan^2 \theta + 1 = \sec^2 \theta$
20. $\sin(2\theta) = 2 \sin \theta \, \cos \theta$
21. $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$

1. [8 marks] Find the exact value of each of the following integrals:

(a) [4 marks] $\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx$.

Solution: let $u = \sin x$. Then $du = \cos x \, dx$, $\cos^2 x = 1 - \sin^2 x = 1 - u^2$, and

$$\begin{aligned}
\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx &= \int_0^{\pi/2} \sin^2 x \cos^2 x \cos x \, dx \\
&= \int_0^1 u^2(1-u^2) \, du \\
&= \int_0^1 (u^2 - u^4) \, du \\
&= \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^1 \\
&= \frac{1}{3} - \frac{1}{5} \\
&= \frac{2}{15}
\end{aligned}$$

(b) [4 marks] $\int_0^{\pi/4} \tan x \sec^4 x \, dx$.

Solution: let $u = \sec x$. Then $du = \sec x \tan x \, dx$, and

$$\begin{aligned}
\int_0^{\pi/4} \tan x \sec^4 x \, dx &= \int_0^{\pi/4} \sec^3 x \tan x \sec x \, dx \\
&= \int_1^{\sqrt{2}} u^3 \, du \\
&= \left[\frac{u^4}{4} \right]_1^{\sqrt{2}} \\
&= 1 - \frac{1}{4} \\
&= \frac{3}{4}
\end{aligned}$$

2. [8 marks] Find $\int x^2 \ln(x^2+1)\,dx$.

Solution: let $u = \ln(x^2+1)$; $dv = x^2\,dx$ and integrate by parts.

$$\begin{aligned}
\int x^2 \ln(x^2+1)\,dx &= uv - \int v\,du \\
&= \frac{x^3}{3}\ln(x^2+1) - \int \frac{x^3}{3}\frac{2x}{x^2+1}\,dx \\
&= \frac{x^3}{3}\ln(x^2+1) - \frac{2}{3}\int \frac{x^4}{x^2+1}\,dx \\
&= \frac{x^3}{3}\ln(x^2+1) - \frac{2}{3}\int \left(x^2 - 1 + \frac{1}{x^2+1}\right)dx \\
&= \frac{x^3}{3}\ln(x^2+1) - \frac{2}{3}\left(\frac{x^3}{3} - x + \tan^{-1}x\right) + C \\
&= \frac{x^3}{3}\ln(x^2+1) - \frac{2x^3}{9} + \frac{2x}{3} - \frac{2\tan^{-1}x}{3} + C
\end{aligned}$$

3. [8 marks] Find the exact value of $\displaystyle\int_0^2 x^2\sqrt{4-x^2}\,dx$.

Solution: let $x = 2\sin\theta$; then $dx = 2\cos\theta\,d\theta$ and

$$\begin{aligned}
\int_0^2 x^2\sqrt{4-x^2}\,dx &= \int_0^{\pi/2} 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,(2\cos\theta)\,d\theta \\
&= 4\int_0^{\pi/2} 4\sin^2\theta\cos^2\theta\,d\theta \\
&= 4\int_0^{\pi/2} (2\sin\theta\cos\theta)^2\,d\theta \\
&= 4\int_0^{\pi/2} \sin^2(2\theta)\,d\theta \\
&= 4\int_0^{\pi/2} \frac{1-\cos(4\theta)}{2}\,d\theta \\
&= 4\left[\frac{\theta}{2} - \frac{\sin(4\theta)}{8}\right]_0^{\pi/2} \\
&= \pi
\end{aligned}$$

4. [8 marks] Find $\displaystyle\int \frac{x^3+4x^2-8x+9}{x(x+1)(x^2+9)}\,dx$.

Solution: use the method of partial fractions. Let

$$\frac{x^3+4x^2-8x+9}{x(x+1)(x^2+9)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+9}.$$

Then

$$\begin{aligned} x^3+4x^2-8x+9 &= A(x+1)(x^2+9) + Bx(x^2+9) + (Cx+D)(x^2+x) \\ &= (A+B+C)x^3 + (A+C+D)x^2 + (9A+9B+D)x + 9A \end{aligned}$$

So $A=1$; and consequently

$$B+C=0, C+D=3, 9B+D=-17$$

Subtracting the first two equations gives

$$B-D=-3.$$

Adding this to the third equation gives

$$10B=-20 \Leftrightarrow B=-2.$$

Then it follows that $C=2$ and $D=1$.

So

$$\begin{aligned} \int \frac{x^3+4x^2-8x+9}{x(x+1)(x^2+9)}\,dx &= \int \frac{1}{x}\,dx - \int \frac{2}{x+1}\,dx + \int \frac{2x+1}{x^2+9}\,dx \\ &= \int \frac{1}{x}\,dx - \int \frac{2}{x+1}\,dx + \int \frac{2x}{x^2+9}\,dx + \int \frac{dx}{x^2+9} \\ &= \ln|x| - 2\ln|x+1| + \ln(x^2+9) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C \end{aligned}$$

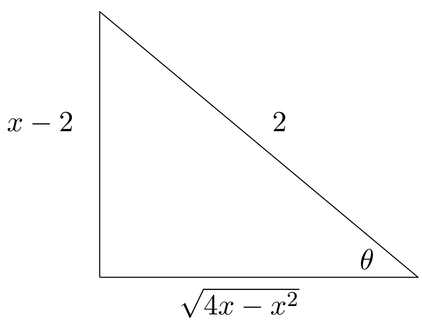
5. [7 marks] Find $\displaystyle\int \frac{x+1}{\sqrt{4x-x^2}}\,dx$.

Solution: complete the square and use a trigonometric substitution.

$$4x - x^2 = 4 - 4 + 4x - x^2 = 4 - (x-2)^2$$

Let $x - 2 = 2\sin\theta$, then $x = 2 + 2\sin\theta$ and

$$\begin{aligned}
\int \frac{x+1}{\sqrt{4x-x^2}}\,dx &= \int \frac{3+2\sin\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta \\
&= \int (3+2\sin\theta)\,d\theta \\
&= 3\theta - 2\cos\theta + C \\
&= 3\sin^{-1}\left(\frac{x-2}{2}\right) - 2\frac{\sqrt{4x-x^2}}{2} + C \\
&= 3\sin^{-1}\left(\frac{x-2}{2}\right) - \sqrt{4x-x^2} + C
\end{aligned}$$



where we used the triangle to the left with

$$\sin\theta = \frac{x-2}{2}$$

to get

$$\cos\theta = \frac{\sqrt{4x-x^2}}{2}.$$

6. [7 marks] Find the value of the following improper integrals.

(a) [3 marks] $\int_0^\infty e^{-ax}\,dx$, for $a > 0$.

Solution:

$$\begin{aligned}
\int_0^\infty e^{-ax}\,dx &= \lim_{N\to\infty}\int_0^N e^{-ax}\,dx \\
&= \lim_{N\to\infty}\left[\frac{e^{-ax}}{-a}\right]_0^N \\
&= \lim_{N\to\infty}\left(-\frac{1}{ae^{aN}}\right)+\frac{1}{a} \\
\text{(since } a>0\text{)} &= 0+\frac{1}{a} \\
&= \frac{1}{a}
\end{aligned}$$

(b) [4 marks] $\int_0^\infty \frac{dx}{x^2+b^2}$, for $b > 0$.

Solution:

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^2+b^2} &= \lim_{N\to\infty}\int_0^N \frac{dx}{x^2+b^2} \\
&= \lim_{N\to\infty}\left[\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)\right]_0^N \\
&= \lim_{N\to\infty}\frac{1}{b}\tan^{-1}\left(\frac{N}{b}\right)-0 \\
\text{(since } b>0\text{)} &= \frac{1}{b}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{2b}
\end{aligned}$$

7. [7 marks] Find $\int \cos(\ln\sqrt{x})\,dx$.

Solution: use integration by parts, twice. First let $u = \cos(\ln\sqrt{x})$ and $dv = dx$. Then

$$\begin{aligned}
\int \cos(\ln x)\,dx &= uv - \int v\,du \\
&= x\cos(\ln\sqrt{x}) - \int x\left(-\frac{\sin(\ln\sqrt{x})}{\sqrt{x}}\right)\frac{dx}{2\sqrt{x}} \\
&= x\cos(\ln\sqrt{x}) + \frac{1}{2}\int \sin(\ln\sqrt{x})\,dx \\
(\text{let } s = \sin(\ln\sqrt{x});\, dt = dx) &= x\cos(\ln\sqrt{x}) + \frac{1}{2}[st - \int t\,ds] \\
&= x\cos(\ln\sqrt{x}) + \frac{x\sin(\ln\sqrt{x})}{2} - \frac{1}{2}\int x\left(\frac{\cos(\ln\sqrt{x})}{\sqrt{x}}\right)\frac{dx}{2\sqrt{x}} \\
&= x\cos(\ln x) + \frac{x\sin(\ln\sqrt{x})}{2} - \frac{1}{4}\int \cos(\ln\sqrt{x})\,dx \\
\Rightarrow \frac{5}{4}\int \cos(\ln\sqrt{x})\,dx &= x\cos(\ln x) + \frac{x\sin(\ln\sqrt{x})}{2} + C \\
\Rightarrow \int \cos(\ln\sqrt{x})\,dx &= \frac{4x\cos(\ln\sqrt{x})}{5} + \frac{2x\sin(\ln\sqrt{x})}{5} + C
\end{aligned}$$

Alternate Solution: since $\ln\sqrt{x} = \frac{1}{2}\ln x$, you could simplify the problem a bit by rewriting it as

$$\int \cos(\frac{1}{2}\ln x)\,dx;$$

but you would still have to use parts twice.

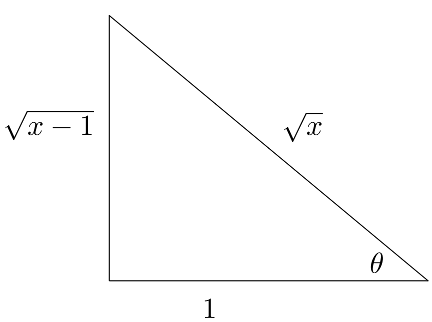
8. [7 marks] Find $\int \frac{dx}{x^3\sqrt{x-1}}$.

Solution: let $u = \sqrt{x-1}$; then $du = \frac{dx}{2\sqrt{x-1}}$ and $x = u^2+1$. So

$$\int \frac{dx}{x^3\sqrt{x-1}} = \int \frac{2\,du}{(u^2+1)^3}$$

Now let $u = \tan\theta$, so $du = \sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{2\,du}{(u^2+1)^3} &= \int \frac{2\sec^2\theta}{(\tan^2\theta+1)^3}\,d\theta \\
&= \int \frac{2\sec^2\theta}{(\sec^2\theta)^3}\,d\theta \\
&= \int 2\cos^4\theta\,d\theta \\
\text{(using Formula 17 with } n=4) &= 2\left(\frac{1}{4}\sin\theta\cos^3\theta + \frac{3}{4}\int \cos^2\theta\,d\theta\right) \\
&= \frac{1}{2}\sin\theta\cos^3\theta + \frac{3}{2}\int \cos^2\theta\,d\theta \\
\text{(using Formula 17 with } n=2) &= \frac{1}{2}\sin\theta\cos^3\theta + \frac{3}{2}\left(\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\int d\theta\right) \\
&= \frac{1}{2}\sin\theta\cos^3\theta + \frac{3}{4}\sin\theta\cos\theta + \frac{3}{4}\theta + C \\
&= \frac{1}{2}\frac{\sqrt{x-1}}{\sqrt{x}}\frac{1}{\sqrt{x^3}} + \frac{3}{4}\frac{\sqrt{x-1}}{\sqrt{x}}\frac{1}{\sqrt{x}} + \frac{3}{4}\tan^{-1}\sqrt{x-1} + C \\
&= \frac{1}{2}\frac{\sqrt{x-1}}{x^2} + \frac{3}{4}\frac{\sqrt{x-1}}{x} + \frac{3}{4}\tan^{-1}\sqrt{x-1} + C,
\end{aligned}$$



where we used the triangle to the left with

$$\sqrt{x-1} = u = \tan\theta$$

to get

$$\sin\theta = \frac{\sqrt{x-1}}{\sqrt{x}} \text{ and } \cos\theta = \frac{1}{\sqrt{x}}.$$