University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, JUNE, 2011**
First Year - CHE, CIV, IND, LME, MEC, MSE

MAT187H1F - CALCULUS II
Exam Type: A

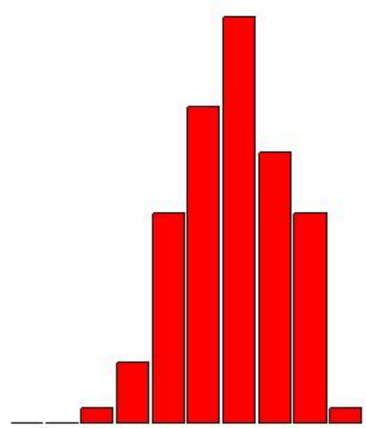
INSTRUCTIONS: Attempt all 9 questions; present your solutions in the exam books provided. You must show your work and give complete explanations to get full marks. The value of each question is indicated in parentheses beside the question number. Only aids permitted: Casio 260, Sharp 520 or TI 30 calculator. **Total Marks:** 100.

Comments:

1. Question 1(b) can also be done by separating variables:
$$\int \frac{dy}{1+y} = \int 2x\,dx \Rightarrow \ln|1+y| = x^2 + C.$$
2. Question 7 was the shortest question on the exam. Most students could find $\frac{d\mathbf{r}}{dt}$, but then didn't know what to do with it.
3. About 20 students, unbelievably, thought $(a+b)^2 = a^2 + b^2$.
4. In Question 4, if you integrate from $\pi/3$ to $5\pi/3$ you are getting the area inside the outer loop.

Breakdown of Results: 100 registered students wrote this exam. The marks ranged from 21% to 93%, and the average was 62.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	1%
A	15%	80-89%	14%
B	18%	70-79%	18%
C	27%	60-69%	27%
D	21%	50-59%	21%
F	19%	40-49%	14%
		30-39%	4%
		20-29%	1%
		10-19%	0%
		0-9%	0%



1. [15 marks; 5 for each part.] Solve the following initial value problems for y as a function of x:

(a) DE: $\cos^2 x \, \dfrac{dy}{dx} = \dfrac{\sin x}{e^y}$; IC: $y(0) = 1$.

Solution: separate variables.

$$\int e^y \, dy = \int \tan x \sec x \, dx \Rightarrow e^y = \sec x + C.$$

Using the initial condition: $y(0) = 1 \Rightarrow e = 1 + C \Rightarrow C = e - 1$. So

$$y = \ln\left(\sec x + e - 1\right).$$

(b) DE: $\dfrac{dy}{dx} - 2x\, y = 2x$; IC: $y(0) = 3$.

Solution:

$$\begin{aligned} \mu = e^{-\int 2x\,dx} = e^{-x^2} \Rightarrow y &= \frac{1}{\mu}\int \mu \, 2x \, dx \\ &= e^{x^2}\left(\int 2x\, e^{-x^2}\, dx\right) \\ &= e^{x^2}\left(-e^{-x^2} + C\right) \\ &= -1 + C\, e^{x^2}. \end{aligned}$$

Now $y(0) = 3 \Rightarrow 3 = -1 + C \Rightarrow C = 4$, so $y = -1 + 4\, e^{x^2}$.

(c) DE: $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$; IC: $y(0) = 2$; $y'(0) = -5$.

Solution: $r^2 + 6r + 9 = 0 \Rightarrow r = -3$, repeated, so $y = A\, e^{-3x} + Bx\, e^{-3x}$, and hence $y' = -3A\, e^{-3x} - 3Bx\, e^{-3x} + B\, e^{-3x}$. Then

$$\left.\begin{array}{rcr} y(0) = 2 & \Rightarrow & 2 = A \\ y'(0) = -5 & \Rightarrow & -5 = -3A + B \end{array}\right\} \Rightarrow \quad A = 2, \; B = 1$$

So

$$y = 2\, e^{-3x} + x\, e^{-3x}.$$

2. [15 marks; 5 for each part.]

(a) What is the interval of convergence of the power series $\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k (x+5)^k$?

Solution: this is an infinite geometric series and converges if and only if

$$\left|\frac{3(x+5)}{4}\right| < 1 \Leftrightarrow |x+5| < \frac{4}{3} - \frac{4}{3} < x+5 < \frac{4}{3} \Leftrightarrow -\frac{19}{3} < x < -\frac{11}{3}.$$

So the open interval of convergence is

$$\left(-\frac{19}{3}, -\frac{11}{3}\right).$$

(b) Find the first three non-zero terms in the Maclaurin series for $\frac{\tan^{-1} x}{1+x}$.

Solution:

$$\begin{aligned}
\frac{\tan^{-1} x}{1+x} &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)(1 - x + x^2 - x^3 + \cdots) \\
&= x - x^2 + x^3 - \frac{x^3}{3} + \cdots \\
&= x - x^2 + \frac{2x^3}{3} + \cdots
\end{aligned}$$

(c) Determine whether the infinite series

$$\sum_{k=1}^{\infty}(-1)^k \frac{k^3}{e^k}$$

converges conditionally, converges absolutely, or diverges.

Solution: use the ratio test for absolute convergence.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left(\frac{(k+1)^3}{k^3}\frac{e^k}{e^{k+1}}\right) = \frac{1}{e} < 1,$$

so the series converges absolutely.

3. [10 marks] Find and classify all the critical points of the function

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

Solution: Let $z = f(x,y)$. Then

$$\frac{\partial z}{\partial x} = 4x^3 - 4y \text{ and } \frac{\partial z}{\partial y} = 4y^3 - 4x.$$

Critical points:

$$\left.\begin{array}{rcl} 4x^3 - 4y & = & 0 \\ 4y^3 - 4x & = & 0 \end{array}\right\} \Rightarrow \left.\begin{array}{rcl} y & = & x^3 \\ x & = & y^3 \end{array}\right\} \Rightarrow x = y = 0 \text{ or } x^8 = 1 \Leftrightarrow x = \pm 1.$$

So the three critical points are

$$(x,y) = (0,0) \text{ and } (x,y) = (1,1) \text{ and } (x,y) = (-1,-1).$$

Second Derivative Test:

$$\frac{\partial^2 z}{\partial x^2} = 12x^2,\ \frac{\partial^2 z}{\partial y^2} = 12y^2,\ \frac{\partial^2 z}{\partial x \partial y} = -4$$

and

$$\Delta = 144x^2y^2 - 16.$$

At both critical points $(x,y) = (1,1)$ and $(-1,-1)$,

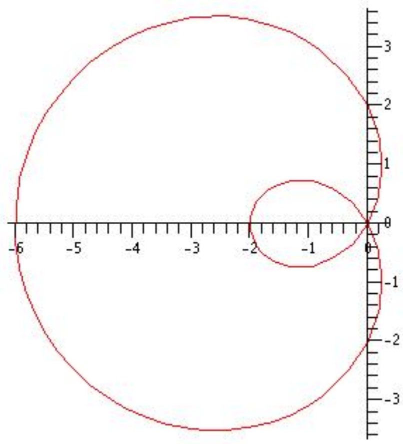
$$\Delta = 128 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = 12 > 0,$$

so f has a minimum value (of $z = -1$) at both critical points $(x,y) = (1,1)$ and $(-1,-1)$.

At the critical point $(x,y) = (0,0)$, $\Delta = -16 < 0$, so there is a saddle point at $(0,0,1)$.

4. [10 marks] Find the area of the region enclosed by the inner loop of the limaçon with polar equation $r = 2 - 4\cos\theta$

Solution:



To find the limits of the inner loop, set $r = 0$:

$$r = 0 \Leftrightarrow \cos\theta = \frac{1}{2} \Leftarrow \theta = \pm\frac{\pi}{3}.$$

To get the area of the inner loop, integrate from $\alpha = -\pi/3$ to $\beta = \pi/3$, or double the integral from $\alpha = 0$ to $\beta = \pi/3$. NB: since $r \leq 0$ along the inner loop, its in quadrants II and III.

$$\begin{aligned}
A &= 2\left(\frac{1}{2}\int_0^{\pi/3} r^2\,d\theta\right) \\
&= \int_0^{\pi/3} (4 - 16\cos\theta + 16\cos^2\theta)\,d\theta \\
\text{(use double angle formula)} \quad &= \int_0^{\pi/3} (4 - 16\cos\theta + 8 + 8\cos(2\theta))\,d\theta \\
&= \int_0^{\pi/3} (12 - 16\cos\theta + 8\cos(2\theta))\,d\theta \\
&= [12\,\theta - 16\sin\theta + 4\sin(2\theta)]_0^{\pi/3} \\
&= 4\pi - 6\sqrt{3}
\end{aligned}$$

5. [10 marks] Find all the critical points on the following curves:

(a) [4 marks] The curve with parametric equations $x = t^2 + 8t$, $y = t^3 - 12t$.

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 12}{2t + 8};$$

so $\dfrac{dy}{dt} = 0 \Leftrightarrow t = \pm 2$ and $\dfrac{dx}{dt} = 0 \Leftrightarrow t = -4$. The critical points are:

$$(x, y) = (-16, -16), (-12, 16), (20, -16).$$

(b) [6 marks] The cardioid with polar equation $r = 1 - \cos\theta$.

Solution: $x = r\cos\theta = (1 - \cos\theta)\cos\theta$; $y = r\sin\theta = (1 - \cos\theta)\sin\theta$; so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin^2\theta + \cos\theta - \cos^2\theta}{\sin\theta\cos\theta - \sin\theta + \sin\theta\cos\theta} = \frac{1 + \cos\theta - 2\cos^2\theta}{2\sin\theta\cos\theta - \sin\theta}.$$

$$\begin{aligned} \frac{dy}{d\theta} = 0 \quad &\Leftrightarrow \quad 2\cos^2\theta - \cos\theta - 1 = 0 \\ &\Leftrightarrow \quad (2\cos\theta + 1)(\cos\theta - 1) = 0 \\ &\Leftrightarrow \quad \cos\theta = -\frac{1}{2} \text{ or } \cos\theta = 1 \\ &\Leftarrow \quad \theta = \pm\frac{2\pi}{3} \text{ or } \theta = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{dx}{d\theta} = 0 \quad &\Leftrightarrow \quad 2\sin\theta\cos\theta - \sin\theta = 0 \\ &\Leftrightarrow \quad \sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2} \\ &\Leftarrow \quad \theta = 0,\ \pi \text{ or } \pm\frac{\pi}{3}. \end{aligned}$$

The six critical points are:

$$(r, \theta) = (0, 0),\ (2, \pi),\ \left(\frac{3}{2}, \pm\frac{2\pi}{3}\right),\ \left(\frac{1}{2}, \pm\frac{\pi}{3}\right);$$

$$\text{or } (x, y) = (0, 0),\ (-2, 0),\ \left(-\frac{3}{4}, \pm\frac{3\sqrt{3}}{4}\right),\ \left(\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right).$$

6. [10 marks] Approximate

$$\int_0^{0.5} \frac{dx}{\sqrt[4]{1+x^2}}$$

to within 10^{-5}, and explain why your approximation is correct to within 10^{-5}.

Solution: use the binomial series

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots$$

with $z = x^2$ and $\alpha = -1/4$. Then

$$\begin{aligned}
&\int_0^{0.5} (1+x^2)^{-1/4}\,dx \\
= \;& \int_0^{0.5} \left(1 - \frac{1}{4}x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}(x^2)^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}(x^2)^3 + \cdots\right) dx \\
= \;& \int_0^{0.5} \left(1 - \frac{1}{4}x^2 + \frac{5}{32}x^4 - \frac{15}{128}x^6 + \frac{195}{2048}x^8 - \frac{663}{8192}x^{10} + \cdots\right) dx \\
= \;& \left[x - \frac{x^3}{12} + \frac{x^5}{32} - \frac{15x^7}{896} + \frac{65x^9}{6144} - \frac{663x^{11}}{90112} + \cdots\right]_0^{0.5} \\
\simeq \;& 0.5 - 0.010416666 + 0.000976562 - 0.000130789 + 0.000020662 \\
\simeq \;& 0.490449769
\end{aligned}$$

which is correct to within

$$\frac{663 \times (0.5)^{11}}{90112} \simeq 0.000003592 < 10^{-5}$$

by the alternating series remainder term. You *must* refer to the alternating series remainder term to get full marks.

7. [10 marks] Find parametric equations of the tangent line to the graph of

$$\mathbf{r}(t) = t^2\,\mathbf{i} - \frac{1}{1+t}\,\mathbf{j} + (4 - t^2)\,\mathbf{k}$$

at the point $(x, y, z) = (4, 1, 0)$.

Solution: at the point $(x, y, z) = (4, 1, 0)$, $t^2 = 4$ and $1 + t = -1$, so $t = -2$. Now

$$\frac{d\mathbf{r}}{dt} = 2t\,\mathbf{i} + \frac{1}{(1+t)^2}\,\mathbf{j} - 2t\,\mathbf{k},$$

which for $t = -2$ is the direction vector

$$-4\,\mathbf{i} + \mathbf{j} + 4\,\mathbf{k}$$

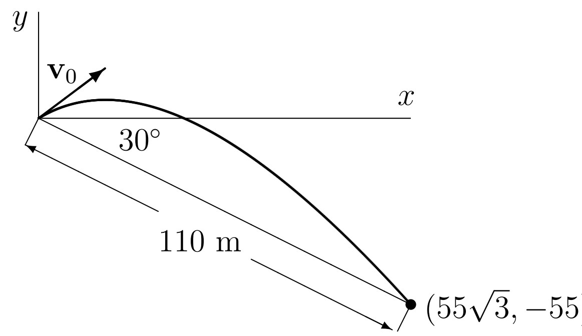
of the tangent line. Thus parametric equations for the tangent line are

$$x = 4 - 4s,\ \ y = 1 + s,\ \ z = 4s,$$

where s is a parameter.

8. [10 marks]

A ski jumper leaves the ramp of the 100 meter ski jump at $t = 0$ with initial velocity $\mathbf{v}_0$ and lands 110 meters down the hill 5 seconds later. Find the jumper's initial velocity if the hill declines at an angle of 30° as measured from the horizontal. For simplicity of calculation, assume the acceleration due to gravity is 10 m/sec^2, and ignore air resistance. In the diagram at the right the origin $(0, 0)$ is where the jumper leaves the ramp.



Solution: Let α be the angle the initial velocity vector $\mathbf{v}_0$ makes with the x-axis; let $v_0 = \|\mathbf{v}_0\|$ be the initial speed. We have: $\mathbf{a} = -10\,\mathbf{j}; \mathbf{v_0} = v_0 \cos\alpha\,\mathbf{i} + v_0 \sin\alpha\,\mathbf{j}$.

So

$$\mathbf{v} = \int \mathbf{a}\,dt = -10t\,\mathbf{j} + \mathbf{v_0} = v_0 \cos\alpha\,\mathbf{i} + (v_0 \sin\alpha - 10t)\,\mathbf{j}$$

and

$$\mathbf{r} = \int \mathbf{v}\,dt = v_0\,t\,\cos\alpha\,\mathbf{i} + (v_0\,t\,\sin\alpha - 5t^2)\,\mathbf{j}.$$

That is,

$$x = v_0\,t\,\cos\alpha \text{ and } y = v_0\,t\,\sin\alpha - 5t^2.$$

When the skier lands on the hill, $t = 5$ and

$$x = 110\cos 30^\circ = 55\sqrt{3},\;\; y = -110\sin 30^\circ = -55.$$

Thus

$$v_0 \cos\alpha = 11\sqrt{3} \text{ and } v_0 \sin\alpha = -11 + 25 = 14.$$

Solving gives

$$\tan\alpha = \frac{14}{11\sqrt{3}} \Rightarrow \alpha \simeq 36.3^\circ$$

and

$$v_0^2 = (11\sqrt{3})^2 + 14^2 = 559 \Rightarrow v_0 \simeq 23.64.$$

The initial velocity is at an angle of 36.3° to the horizontal with speed 23.64 m/sec.

9. [10 marks] Consider the curve with vector equation $\mathbf{r} = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

Solution:

$$\frac{d\mathbf{r}}{dt} = e^t(\cos t - \sin t)\,\mathbf{i} + e^t(\sin t + \cos t)\,\mathbf{j};$$

$$\begin{aligned}
\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2} \\
&= e^t\sqrt{\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t} \\
&= e^t\sqrt{1+1} \\
&= e^t\sqrt{2}
\end{aligned}$$

(b) [4 marks] Find an arc length parametrization of the above curve, with reference point $(1, 0)$, for which $t = 0$.

Solution:

$$s = \int_0^t \left\|\frac{d\mathbf{r}}{du}\right\| du = \left[e^u\sqrt{2}\right]_0^t = \sqrt{2}\,e^t - \sqrt{2}$$

So

$$e^t - 1 = \frac{s}{\sqrt{2}} \Leftrightarrow t = \ln\left(\frac{s}{\sqrt{2}} + 1\right).$$

Consequently an arc length parametrization of the curve is

$$\mathbf{r} = \left(\frac{s}{\sqrt{2}} + 1\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right)\mathbf{i} + \left(\frac{s}{\sqrt{2}} + 1\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right)\mathbf{j}.$$