MAT187 - Calculus II - Winter 2015

Term Test 2 - March 10, 2015

Time allotted: 100 minutes. Aids permitted: None.

Total marks: 50

Full Name: ______________________________
Last First

Student Number: ______________________________

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages (including this title page). Make sure you have all of them.
- You can use pages 12–14 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 12–14.

GOOD LUCK!

PART I No explanation is necessary. **(10 marks)**

1. Consider the differential equation

$$y' = (y^2 - 3y + 2)3t^2.$$

Write this equation in separable form:

$$\int \underline{\hspace{6cm}}\, dy = \int \underline{\hspace{6cm}}\, dt.$$

2. Consider the separable differential equation

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dt} = 1.$$

Then

$$y(t) = \underline{\hspace{10cm}}.$$

3. Consider the differential equation

$$y' + \tan(t)y = \cos(t).$$

What is the integrating factor $\mu(t)$?

$$\mu(t) = \underline{\hspace{10cm}}.$$

4. Consider the differential equation

$$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}.$$

The integrating factor is $\mu(t) = t^2$. Then the general solution is

$$y(t) = \underline{\hspace{10cm}}.$$

5. Consider the differential equation

$$t^2y'' + 7ty' + 9y = 0.$$

If we look for a solution of the form $y = t^p$, then

$$p = \underline{\hspace{10cm}}$$

6. Circle the correct option. The series $\displaystyle\sum_{k=33}^{\infty} \frac{(-1)^k}{k^2}$

(a) converges absolutely. **(b)** converges conditionally. **(c)** diverges.

7. Consider the divergent series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

We want to add the first N terms: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N}$ to make sure that we obtain a sum larger than 42.

Then we need:

$N \geqslant$ ____________________________________

8. Consider the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$$

We can approximate the series by adding the first N terms: $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \cdots + \frac{(-1)^{N+1}}{\sqrt{N}}$.

To make sure that the error is smaller than $\frac{1}{1000}$, we need

$N \geqslant$ ____________________________________

9. Recall that when we approximate a function $f(x)$ by $p_n(x)$, the Taylor polynomial of degree n centered at $a = 0$, then the remainder is

$$f(x) - p_n(x) = R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1},$$

where c is a point between x and 0.

When we approximate $f(x) = e^{3x}$ at $x = 1$ by $p_4(1) = \frac{131}{8}$, the error we make is

error $\leqslant$ ____________________________________

(your answer should not depend on n, x or c)

10. Consider the differential equation

$$y' = y^2 - (t-4)^3.$$

If $y(4) = 0$, then the solution $y(t)$ has

(a) a relative minimum at $t = 4$.

(b) a relative maximum at $t = 4$.

(c) an inflection point at $t = 4$.

(d) none of the other options.

PART II **Justify** your answers.

11. Consider a population of jelly fish which satisfy the following growth model: **(10 marks)**

$$P'(t) = r\big(P(t) - T\big)\big(P(t) - K\big)^2 \qquad \text{where } 0 < T < K.$$

Initially, the population is $P_0 \geqslant 0$.

(a) (2 marks) What are the equilibrium solutions?

Answer :

(b) (2 marks) For which value or values of P_0 will the population grow without bound?

Answer : $P_0 \in$

(c) (2 marks) For which value or values of P_0 will the population become extinct?

Answer : $P_0 \in$

(d) (1 mark) For the values of P_0 you found in **(c)**, will the population become extinct in a finite amount of time?

Answer : (Circle the correct option)	**Yes**	**No**

(e) (3 marks) There are several different types of behaviour for $P(t)$ depending on the initial value P_0. Sketch what each of these types would look like.

12. Let $r, K > 0$. Find the solution of **(5 marks)**

$$K\frac{dP}{dt} = rP(K-P)$$
$$P(0) = \frac{K}{2}$$

You can assume that the solution satisfies $0 \leqslant P(t) \leqslant K$.

Answer : $P(t) =$

13. Let $r > 0$. Examine the following series for convergence: **(4 marks)**

$$\sum_{k=1}^{\infty} \frac{k^k}{k!\, r^k}.$$

Fill in the space below and justify your answer. Don't worry about the boundary point.

Answer : Series converges for $\boldsymbol{r} >$ ____________________

Series diverges for $\boldsymbol{r} <$ ____________________

14. Consider the function $f(x) = e^x \sin(x)$. **(6 marks)**

(a) (3 marks) Find the Taylor polynomial of degree 3 to approximate $f(x)$ near $x = 0$.

Answer : $p_3(x) =$

(b) (3 marks) Using part **(a)**, we can approximate $e^{\frac{\pi}{2}}$ by $p_3\left(\frac{\pi}{2}\right)$. Give an upper bound for the error of this approximation. You can use the formula from question **9**.

Answer : error $\leqslant$

15. A ball is bouncing on the ground on a planet with gravitational constant g. **(10 marks)**

Assume that it takes the same time for the ball to go from the ground up to a height h as it takes to drop from a height h to the ground. The time for each of these is $\sqrt{\frac{2h}{g}}$ seconds.

Each time the ball bounces to $\frac{4}{9}$ of the height of the previous bounce.

Initially it is dropped from a height of 1 metre.

(a) (4 marks) Find the total distance travelled by the ball.

Answer : Total distance =

(b) (4 marks) Let T_n be the total elapsed time it takes from the beginning when the ball is dropped until the ball hits the floor for the n^{th} time. Find a formula for T_n.

Answer : $T_n =$

(c) (2 marks) Does the ball ever stop bouncing? If so, how long does it take?

Answer :

(**Bonus**) What is the average speed of the ball? **(2 marks)**

Answer : Average Speed $=$

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The end.