University of Toronto
Solutions to **MAT 187H1F TERM TEST**
of **FRIDAY, MAY 23, 2008**
Duration: 90 minutes

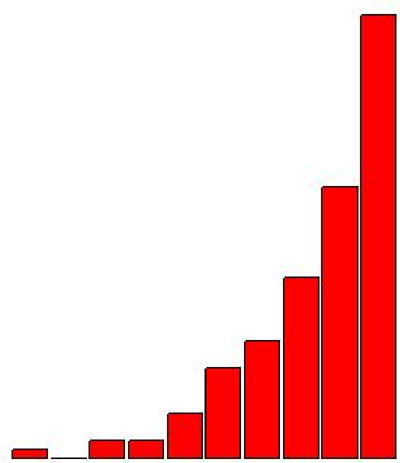
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Answer all seven questions. Present your solutions in the space provided. The value for each question is indicated in parantheses beside the question number. Do not tear any pages from this test. Make sure this test contains 8 pages.

General Comments: this was a test consisting entirely of routine questions. It's nice to see that most of you took advantage of this and did very well. However, although Question 7 was right out of the homework – Section 7.3, #23 – it still caused some problems. And Question 1 required only a simple substitution, but many students seem to have missed it.

Breakdown of Results: 136 registered students wrote this test. The marks ranged from 8.3% to 100%, and the average was 78.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	36.0%
A	58.1%	80-89%	22.1%
B	17.6%	70-79%	17.6%
C	9.6%	60-69%	9.6%
D	7.4%	50-59%	7.4%
F	7.4%	40-49%	3.7%
		30-39%	1.5%
		20-29%	1.5%
		10-19%	0.0%
		0-9%	0.7 %



1. [6 marks] Find $\displaystyle\int_0^4 \frac{1}{\sqrt{x}\,(x+1)}\,dx$ (Hint: $x = (\sqrt{x})^2$)

Soluton: let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, and

$$\begin{aligned}
\int_0^4 \frac{1}{\sqrt{x}\,(x+1)}\,dx &= 2\int_0^2 \frac{1}{u^2+1}\,du \\
&= 2\left[\tan^{-1} x\right]_0^2 \\
&= 2\tan^{-1} 2 - 2\tan^{-1} 0 \\
&= 2\tan^{-1} 2
\end{aligned}$$

Note: after substitution the improper integral becomes a proper integral.

2. [8 marks] Find $\displaystyle\int \sin^3 x \, \cos^4 x \, dx$

Soluton: let $u = \cos x$. Then $du = -\sin x \, dx$, and

$$
\begin{aligned}
\int \sin^3 x \, \cos^4 x \, dx &= \int \sin^2 x \cos^4 x \, \sin x \, dx \\
&= \int \left(1 - \cos^2 x\right) \cos^4 x \, \sin x \, dx \\
&= \int (1 - u^2) \, u^4 \, (-du) \\
&= \int \left(u^6 - u^4\right) \, du \\
&= \frac{1}{7} u^7 - \frac{1}{5} u^5 + C \\
&= \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + C
\end{aligned}
$$

3. [10 marks] Find $\displaystyle\int \frac{6x^2-5x+3}{(x^2+1)(x-1)}\,dx$

Soluton: let

$$\frac{6x^2-5x+3}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Then

$$\begin{aligned}
\frac{6x^2-5x+3}{(x^2+1)(x-1)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \\
&= \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)} \\
\Leftrightarrow 6x^2-5x+3 &= (A+B)x^2+(C-B)x+A-C
\end{aligned}$$

Solve

$$\left\{\begin{array}{ccccccc} A & + & B & & & = & 6 \\ & - & B & + & C & = & -5 \\ A & & & - & C & = & 3 \end{array}\right. \Leftrightarrow (A,B,C)=(2,4,-1).$$

So

$$\begin{aligned}
\int \frac{6x^2-5x+3}{(x^2+1)(x-1)}\,dx &= \int \left(\frac{2}{x-1} + \frac{4x-1}{x^2+1}\right) dx \\
&= 2\int \frac{1}{x-1}\,dx + 2\int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx \\
&= 2\ln|x-1| + 2\ln(x^2+1) - \tan^{-1} x + C
\end{aligned}$$

4. [10 marks] Find $\displaystyle\int \frac{x+1}{\sqrt{4x-x^2}}\,dx$

Soluton: complete the square and use a trigonometric substitution.

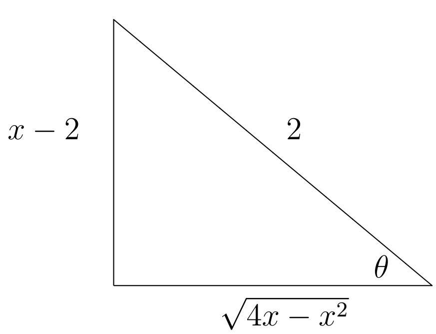
$$\begin{aligned}
4x - x^2 &= -(x^2-4x) \\
&= -(x^2-4x+4-4) \\
&= -\left(x^2-4x+4\right)+4 \\
&= -\left(x-2\right)^2+4 \\
&= 2^2-\left(x-2\right)^2
\end{aligned}$$

Let

$$x-2 = 2\sin\theta;$$

then, putting everything in terms of θ :

$$\begin{aligned}
\int \frac{x+1}{\sqrt{4x-x^2}}\,dx &= \int \frac{x+1}{\sqrt{2^2-(x-2)^2}}\,dx \\
&= \int \frac{2\sin\theta+2+1}{\sqrt{2^2-2^2\sin^2\theta}}\,2\cos\theta\,d\theta \\
&= \int \frac{2\sin\theta+3}{\sqrt{1-\sin^2\theta}}\,\cos\theta\,d\theta \\
&= \int (2\sin\theta+3)\,d\theta \\
&= -2\cos\theta+3\theta+C
\end{aligned}$$



In the triangle, $\sin\theta = \dfrac{x-2}{2}$. Then

$$\cos\theta = \frac{\sqrt{4x-x^2}}{2}.$$

So

$$\begin{aligned}
\int \frac{x+1}{\sqrt{4x-x^2}}\,dx\,dx &= -2\cos\theta+3\theta+C \\
&= -2\frac{\sqrt{4x-x^2}}{2}+3\sin^{-1}\left(\frac{x-2}{2}\right)+C \\
&= -\sqrt{4x-x^2}+3\sin^{-1}\left(\frac{x-2}{2}\right)+C
\end{aligned}$$

5. [10 marks] Find $\displaystyle\int_2^{\infty} \frac{\sqrt{x^2-4}}{x^3}\,dx$

Soluton: in this solution, changing the limits as the trig substitution is made, turns the improper integral into a proper integral.

Let $x = 2\sec\theta$; then $dx = 2\sec\theta\tan\theta\,d\theta$ and as

$$x \longrightarrow \infty, \theta \longrightarrow \frac{\pi}{2}:$$

so

$$\begin{aligned}
\int_2^{\infty} \frac{\sqrt{x^2-4}}{x^3}\,dx &= \int_0^{\pi/2} \frac{\sqrt{4\sec^2\theta - 4}}{8\sec^3\theta} 2\sec\theta\tan\theta\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\tan\theta}{\sec^3\theta} \sec\theta\tan\theta\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\tan^2\theta}{\sec^2\theta}\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,d\theta \\
&= \frac{1}{2}\int \frac{1-\cos(2\theta)}{2}\,d\theta \\
&= \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2} \\
&= \frac{\pi}{8}
\end{aligned}$$

6. [8 marks] Find $\int \sin(\ln x)\,dx$

Soluton: use integration by parts, twice.

$$
\begin{aligned}
\int \sin(\ln x)\,dx &= \int u\,dv, \text{ with } u = \sin(\ln x), dv = \,dx \\
&= uv - \int v\,du \\
&= x\sin(\ln x) - \int \frac{\cos(\ln x)}{x}\,x\,dx \\
&= x\sin(\ln x) - \int \cos(\ln x)\,dx \\
&= x\sin(\ln x) - \int s\,dt, \text{ with } s = \cos(\ln x), dt = dx \\
&= x\sin(\ln x) - (st - \int t\,ds) \\
&= x\sin(\ln x) - x\cos(\ln x) + \int -\frac{\sin(\ln x)}{x}\,x\,dx \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx \\
\Rightarrow 2\int \sin(\ln x)\,dx &= x\sin(\ln x) - x\cos(\ln x) + C \\
\Rightarrow \int \sin(\ln x)\,dx &= \frac{1}{2}(x\sin(\ln x) - x\cos(\ln x)) + C'
\end{aligned}
$$

7. [8 marks] Find $\displaystyle\int \sec^{-1}\sqrt{x}\,dx$ (Recall: $\sec^{-1}$ is the same as arcsec.)

Soluton: Use integration by parts, and be careful!

$$\begin{aligned}
\int \sec^{-1}\sqrt{x}\,dx = \int u\,dv &= uv - \int v\,du, \text{ with } u = \sec^{-1}\sqrt{x}, dv = dx \\
&= x\sec^{-1}\sqrt{x} - \int \frac{1}{\sqrt{x}\sqrt{x-1}}\frac{1}{2\sqrt{x}}\,x\,dx \\
&= x\sec^{-1}\sqrt{x} - \frac{1}{2}\int \frac{1}{\sqrt{x-1}}\,dx \\
&= x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C
\end{aligned}$$