University of Toronto

MAT 187H1F TERM TEST

THURSDAY, MAY 16, 2013

Duration: 100 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Present your solutions to the following questions in the booklets provided. The value for each question is indicated in parantheses beside the question number. You may find the sheet of formulas on the back of this page useful. **TOTAL MARKS: 60**

Answers:

1. [6 marks] $\displaystyle\int_0^{\pi/4} \tan^4 x \,\sec^4 x\, dx = \frac{12}{35}$
2. [8 marks] $\displaystyle\int \sin^{-1} x\, dx = x \,\sin^{-1} x + \sqrt{1-x^2} + C$
3. [10 marks] $\displaystyle\int \frac{9x^2+10x+4}{x^4-3x^2-4}\, dx = -\ln|x+2| + 3\ln|x-2| - \ln(x^2+1) + \tan^{-1} x + C$
4. [8 marks] $\displaystyle\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}$
5. [8 marks] $\displaystyle\int \frac{(x+1)\, dx}{\sqrt{x^2-2x}} = 2\ln|x-1+\sqrt{x^2-2x}| + \sqrt{x^2-2x} + C$
6. [6 marks] $\displaystyle\int_3^\infty \frac{2\, dx}{x^2-1} = \ln 2$
7. [6 marks] $\displaystyle\int_0^\infty \frac{x^6\, dx}{1+x^{14}} = \frac{\pi}{14}$
8. [8 marks] $\displaystyle\int \cos(\ln x)\, dx = \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C$

Solutions:

1. Let $u = \tan x$, then $du = \sec^2 x\,dx$ and $\sec^2 x = 1 + \tan^2 x = 1 + u^2$. So

$$\int_0^{\pi/4} \tan^4 x \sec^4 x\,dx = \int_0^1 u^4(1+u^2)\,du = \int_0^1 (u^4+u^6)\,du = \left[\frac{u^5}{5}+\frac{u^7}{7}\right]_0^1 = \frac{12}{35}$$

2. Use parts: let $u = \sin^{-1} x$ and let $dv = dx$. Then

$$du = \frac{1}{\sqrt{1-x^2}}\,dx \text{ and } v = \int dx = x,$$

so

$$\begin{aligned}\int \sin^{-1} x\,dx = \int u\,dv &= uv - \int v\,du \\ &= x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}} \\ &= x\sin^{-1} x + \frac{1}{2}\int \frac{(-2x)}{\sqrt{1-x^2}}\,dx \\ &= x\sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

3. Factor the denominator and then use partial fractions.

$$x^4 - 3x^2 - 4 = (x^2-4)(x^2+1) = (x+2)(x-2)(x^2+1).$$

Let

$$\begin{aligned}\frac{9x^2+10x+4}{x^4-3x^2-4} &= \frac{A}{x+2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+1} \\ &= \frac{A(x-2)(x^2+1) + B(x+2)(x^2+1) + (Cx+D)(x^2-4)}{x^4-3x^2-4} \\ &= \frac{(A+B+C)x^3 + (-2A+2B+D)x^2 + (A+B-4C)x - 2A + 2B - 4D}{x^4-3x^2-4}\end{aligned}$$

Then $\left\{\begin{array}{rcrcrcrcl} A & + & B & + & C & & & = & 0 \\ -2A & + & 2B & & & + & D & = & 9 \\ A & + & B & - & 4C & & & = & 10 \\ -2A & + & 2B & & & - & 4D & = & 4 \end{array}\right. \Rightarrow A = -1, B = 3, C = -2, D = 1.$

So

$$\begin{aligned}\int \frac{9x^2+10x+4}{x^4-3x^2-4}\,dx &= -\int \frac{dx}{x+2} + \int \frac{3\,dx}{x-2} - \int \frac{2x\,dx}{x^2+1} + \int \frac{dx}{x^2+1} \\ &= -\ln|x+2| + 3\ln|x-2| - \ln(x^2+1) + \tan^{-1} x + K\end{aligned}$$

4. Here are two solutions:

Method 1: use a trig substitution. Let $x = \sin\theta$; then $dx = \cos\theta \, d\theta$ and

$$\int_0^1 \sqrt{1-x^2}\, dx = \int_0^{\pi/2} \cos^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} (1+\cos(2\theta))\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/2} = \frac{\pi}{4}.$$

Method 2: the integral represents one quarter of the area of the unit circle with equation $x^2 + y^2 = 1$, so

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}.$$

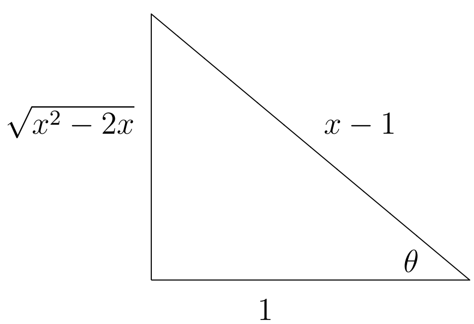
5. Complete the square and use a trig substitution:

$$x^2 - 2x = x^2 - 2x + 1 - 1 = (x-1)^2 - 1,$$

so let $x - 1 = \sec\theta$. Then $x = 1 + \sec\theta$ and $dx = \sec\theta \, \tan\theta \, d\theta$. Consequently

$$\int \frac{(x+1)\, dx}{\sqrt{x^2-2x}} = \int \frac{(\sec\theta + 2)}{\sqrt{\sec^2\theta - 1}} \sec\theta \tan\theta \, d\theta = \int \frac{(\sec\theta+2)}{\tan\theta} \sec\theta\tan\theta \, d\theta =$$

$$\int (\sec^2\theta + 2\sec\theta)\, d\theta = \int \sec^2\theta \, d\theta + 2\int \sec\theta \, d\theta = \tan\theta + 2\ln|\sec\theta + \tan\theta| + C$$



In the triangle to the left,

$$x - 1 = \sec\theta.$$

So

$$\tan\theta = \sqrt{x^2 - 2x}.$$

And finally

$$\int \frac{(x+1)\, dx}{\sqrt{x^2-2x}} = \sqrt{x^2-2x} + 2\ln|x - 1 + \sqrt{x^2-2x}| + C.$$

6. This is an improper integral.

$$\int_3^\infty \frac{2\,dx}{x^2-1} = \lim_{b\to\infty}\int_3^b \frac{2\,dx}{x^2-1} = \lim_{b\to\infty}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx =$$

$$\lim_{b\to\infty}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_3^b = \lim_{b\to\infty}\ln\left|\frac{b-1}{b+1}\right| - \ln\frac{2}{4} = \ln 1 - \ln 2^{-1} = \ln 2.$$

7. This is also an improper integral. Start with $u = x^7$; then $du = 7x^6\,dx$ and

$$\int_0^\infty \frac{x^6\,dx}{1+x^{14}} = \frac{1}{7}\int_0^\infty \frac{du}{1+u^2}\,du = \frac{1}{7}\lim_{b\to\infty}\left[\tan^{-1} u\right]_0^b = \frac{1}{7}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{14}.$$

8. Use integration by parts, twice.

Start with $u = \cos(\ln x), dv = dx$. Then

$$du = -\frac{\sin(\ln x)}{x}\,dx, v = x$$

and so

$$\begin{aligned}\int \cos(\ln x)\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= x\cos(\ln x) - \int x\left(-\frac{\sin(\ln x)}{x}\right)dx \\ &= x\cos(\ln x) + \int \sin(\ln x)\,dx\end{aligned}$$

To integrate $\int \sin(\ln x)\,dx$ let $s = \sin(\ln x),\ dt = dx$. Then

$$ds = \frac{\cos(\ln x)}{x}\,dx \text{ and } t = x$$

so

$$\int \sin(\ln x)\,dx = \int s\,dt = st - \int t\,ds = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

Putting it all together,

$$\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$$

$$\Rightarrow 2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) + C$$

$$\Rightarrow \int \cos(\ln x)\,dx = \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C'.$$