MAT187H1F - Calculus II - Spring 2016

Solutions to Term Test 1 - June 1, 2016

Time allotted: 110 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

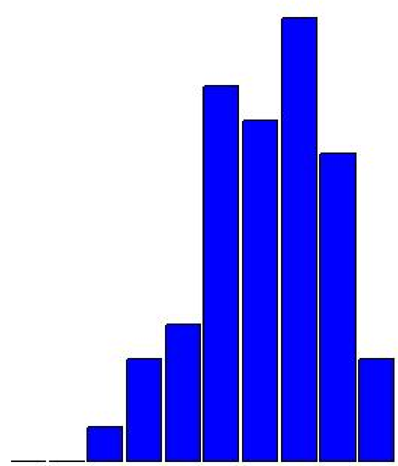
General Comments:

1. Generally speaking the results on this test were good; the average was 66%, which is much higher than the average on the first test last summer, 57%, and also higher than the average of the first test in February 2016, 60.5%.

2. The only questions on this test with failing averages were the last two, Questions 7 and 8. However, only Question 8 was meant as a "challenge." Question 7 was quite routine. I did a very similar example in class; and a very similar example showed up in the 'review' session at the end of Tuesday's class before the test!

Breakdown of Results: 54 students wrote this test. The marks ranged from 22.5% to 98.75%, and the average was 66.25%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.6%
A	22.2%	80-89%	16.6%
B	24.1%	70-79%	24.1%
C	18.5%	60-69%	18.5%
D	20.4%	50-59%	20.4%
F	14.8%	40-49%	7.4%
		30-39%	5.6%
		20-29%	1.8%
		10-19%	0.0 %
		0-9%	0.0%



PART I : No explanation is necessary. Answer all parts of Questions 1 and 2 by putting your answer in the appropriate blank or by circling your choice(s).

Question 1 avg: 7.54/10

1.(a) [2 marks] The exact value of $\displaystyle\int_0^\infty e^{-2x}\,dx$ is: 1/2

$$\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty}\int_0^b e^{-2x}\,dx = \lim_{b\to}\infty\left[-\frac{e^{-2x}}{2}\right]_b^\infty = 0+\frac{1}{2}=\frac{1}{2}.$$

1.(b) [6 marks] Consider the integral $\displaystyle\int_0^{\pi/4}\tan^{37}x\,\sec^{10}x\,dx$. Use a substitution to obtain

$$\int_0^{\pi/4}\tan^{37}x\,\sec^{10}x\,dx = \int_a^b f(u)\,du,$$

where $f(u)$ is a polynomial function of u. Then

		or		
$u =$	$\tan x$		$u =$	$\sec x$
$a =$	0		$a =$	1
$b =$	1		$b =$	$\sqrt{2}$
$f(u) =$	$u^{37}(1+u^2)^4$		$f(u) =$	$u^9(u^2-1)^{18}$

1.(c) [2 marks] A radioactive substance has half-life 14 years. How long will it take for 75% of it to decay?

Quick way: since $25\% = \dfrac{1}{4} = \left(\dfrac{1}{2}\right)^2$, it will take two half-lives until 25% is left.

Answer: 28 yrs

Question 2 avg: 8.17/10

2.(a) [6 marks] Consider the rational function

$$\frac{x^4 - 10x^2 + x - 1}{(x^3 - x^2)(x^2 + x + 1)(x^2 + 4)^2}.$$

To find its partial fraction decompostion we write this function as a sum of the following terms (circle all that apply):

$\boxed{\text{(a)}}$	$\frac{A}{x}$	$\boxed{\text{(b)}}$	$\frac{D}{x-1}$	$\boxed{\text{(c)}}$	$\frac{G+Hx}{x^2+4}$	$\boxed{\text{(d)}}$	$\frac{Mx+N}{x^2+x+1}$	(e)	$\frac{Sx+T}{x^3-x^2}$
$\boxed{\text{(f)}}$	$\frac{B}{x^2}$	(g)	$\frac{E}{(x-1)^2}$	$\boxed{\text{(h)}}$	$\frac{I+Jx}{(x^2+4)^2}$	(i)	$\frac{Ox+P}{(x^2+x+1)^2}$	(j)	$\frac{Ux+V}{(x^3-x^2)^2}$
(k)	$\frac{C}{x^3}$	(l)	$\frac{F}{(x-1)^3}$	(m)	$\frac{K+Lx}{(x^2+4)^3}$	(n)	$\frac{Qx+R}{(x^2+x+1)^3}$	(o)	$\frac{Wx+X}{(x^3-x^2)^3}$

2.(b) [2 marks] Circle all of the following differential equations which are **not** separable.

$$(a)\ \frac{dy}{dx} = \frac{(y^2+1)^3}{x^2-9} \quad (b)\ \frac{dy}{dx} = e^x + e^{x+y} \quad \boxed{\text{(c)}}\ \frac{dy}{dx} = \ln x + \sin y$$

2.(c) [2 marks] Consider the initial value problem

$$\text{DE: } \frac{dP}{dt} = \frac{P}{10} - \frac{P^2}{750}, \quad \text{IC: } P(0) = 5.$$

The value of $\lim_{t\to\infty} P$ is: 75

$$\frac{dP}{dt} = \frac{P}{10} - \frac{P^2}{750} = \frac{P}{10}\left(1 - \frac{P}{75}\right)$$

PART II : Present **complete** solutions to the following questions in the space provided.

3. [10 marks; avg: 9.24/10] Newton's Law of Cooling states that

$$\frac{dT}{dt} = -k(T - A),$$

where T is the temperature of an object at time t, in a room with constant ambient temperature A, and k is a positive constant.

Suppose a cup of coffee, initially at temperature 90 C, is placed on a table in a room with constant temperature 20 C. If the temperature of the coffee is 70 C five minutes later, when will the temperature of the coffee be 40 C? Sketch a graph of T for $t \geq 0$, labeling any horizontal asymptotes.

Solution I: we have $A = 20$ and $T_0 = 90$, so if you memorized the solution you know that

$$T = 20 + 70e^{-kt}.$$

To find k, let $t = 5$ and $T = 70$:

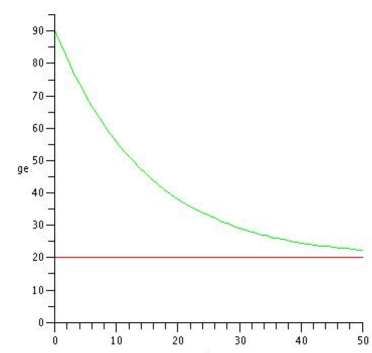
$$70 = 20 + 70e^{-5k} \Leftrightarrow \frac{50}{70} = e^{-5k} \Leftrightarrow e^{5k} = \frac{7}{5} = 1.4 \Leftrightarrow k = \frac{1}{5}\ln 1.4 \approx 0.067294447$$

Now let $T = 40$ and solve for t:

$$40 = 20 + 70e^{-kt} \Leftrightarrow \frac{20}{70} = e^{-kt}$$

$$\Leftrightarrow e^{kt} = 3.5 \Leftrightarrow kt = \ln 3.5 \Leftrightarrow t = \frac{5\ \ln 3.5}{\ln 1.4} \approx 18.616$$

So it will take about 18.6 min for the coffee to reach 40 C.
The graph is to the right. $T = 20$ is a horizontal asymptote.



Solution II: separate variables. NB: it is not necessary to solve for T.

$$\int \frac{dT}{T - 20} = -\int k\, dt \Leftrightarrow \ln(T - 20) = -kt + C.$$

To find C, let $t = 0, T = 90$: $C = \ln 70$. Now to find k, let $t = 5, T = 70$:

$$\ln 50 = -5k + \ln 70 \Leftrightarrow 5k = \ln 70 - \ln 50 \Leftrightarrow k = \frac{1}{5}\ln 1.4.$$

Finally, let $T = 40$ and solve for t:

$$\ln 20 = -\frac{t\ \ln 1.4}{5} + \ln 70 \Leftrightarrow \frac{t \ln 1.4}{5} = \ln 70 - \ln 20 \Leftrightarrow t = \frac{5 \ln 3.5}{\ln 1.4},$$

as before.

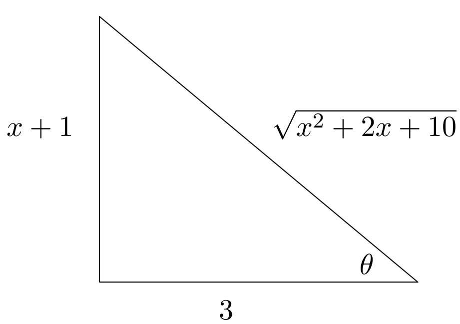
4. [10 marks; avg: 6.89/10] Find $\displaystyle\int \frac{(4+3x)}{\sqrt{x^2+2x+10}}\,dx$.

Solution: complete the square and use a trig substitution.

$$x^2+2x+10 = x^2+2x+1+9 = (x+1)^2+3^2$$

Let $x+1 = 3\tan\theta$. Then $x = 3\tan\theta - 1$ and $dx = 3\sec^2\theta\,d\theta$. So

$$\begin{aligned}
\int \frac{(4+3x)}{\sqrt{x^2+2x+10}}\,dx &= \int \frac{(4+9\tan\theta-3)}{\sqrt{9\tan^2\theta+9}}\,3\sec^2\theta\,d\theta \\
&= \int \frac{9\tan\theta+1}{3\sec\theta}\,3\sec^2\theta\,d\theta \\
&= \int (9\tan\theta\sec\theta+\sec\theta)\,d\theta \\
&= 9\sec\theta + \ln|\sec\theta+\tan\theta| + C \\
&= 3\sqrt{x^2+2x+10} + \ln\left|\frac{\sqrt{x^2+2x+10}+x+1}{3}\right| + C \\
\text{(or equivalently)} &= 3\sqrt{x^2+2x+10} + \ln(\sqrt{x^2+2x+10}+x+1) + C,
\end{aligned}$$



where we used the triangle to the left with

$$\tan\theta = \frac{x+1}{3}$$

to get

$$\sec\theta = \frac{\sqrt{x^2+2x+10}}{3}.$$

5. [avg: 7.31/10] Find the general solution, y in terms of x, for each of the following differential equations:

(a) [5 marks] $\dfrac{dy}{dx} = (1 + y^2)(\sin^2 x)$

Solution: separate variables.

$$\int \frac{dy}{1+y^2} = \int \sin^2 x\, dx = \frac{1}{2}\int (1 - \cos(2x))\, dx$$

$$\Rightarrow \tan^{-1} y = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

$$\Rightarrow y = \tan\left(\frac{x}{2} - \frac{\sin(2x)}{4} + C\right)$$

(b) [5 marks] $\dfrac{dy}{dx} + y \tan x = \sec x$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

Solution: use the method of the integrating factor.

$$\mu = e^{\int \tan x\, dx} = e^{\ln(\sec x)} = \sec x.$$

Then

$$y = \frac{1}{\mu}\int \mu \sec x\, dx = \cos x \int \sec^2 x\, dx = \cos x\,(\tan x + C) = \sin x + C\, \cos x$$

6. [10 marks; avg: 5.76/10] Find the exact value of $\int_1^4 x \sec^{-1}\sqrt{x}\,dx$. (NB: don't forget the chain rule.)

Solution: use integration by parts. Let

$$u = \sec^{-1}\sqrt{x};\;\; dv = x\,dx.$$

Then

$$du = \frac{1}{\sqrt{x}}\frac{1}{\sqrt{(\sqrt{x})^2 - 1}}\left(\frac{1}{2\sqrt{x}}\right)dx = \frac{1}{2}\frac{1}{x\sqrt{x-1}}\,dx; \quad v = \frac{x^2}{2}$$

and

$$\begin{aligned}
\int_1^4 x \sec^{-1}\sqrt{x}\,dx &= [uv]_1^4 - \int_1^4 v\,du \\
&= \left[\frac{x^2}{2}\left(\sec^{-1}\sqrt{x}\right)\right]_1^4 - \frac{1}{4}\int_1^4 \frac{x}{\sqrt{x-1}}\,dx \\
(\text{let } x - 1 = t^2) \quad &= 8\sec^{-1} 2 - \frac{1}{2}\sec^{-1} 1 - \frac{1}{4}\int_0^{\sqrt{3}} \frac{(t^2+1)\,2t}{t}\,dt \\
&= \frac{8\pi}{3} - 0 - \frac{1}{2}\int_0^{\sqrt{3}} (t^2+1)\,dt \\
&= \frac{8\pi}{3} - \frac{1}{2}\left[\frac{t^3}{3} + t\right]_0^{\sqrt{3}} \\
&= \frac{8\pi}{3} - \frac{1}{2}\left(\frac{3\sqrt{3}}{3} + \sqrt{3}\right) \\
&= \frac{8\pi}{3} - \sqrt{3}
\end{aligned}$$

Continued...

7. [10 marks; avg: 4.46/10] Recall: if $x(t)$ is the mass of solute dissolved in a solution of volume $V(t)$, at time t, in a large mixing tank, then

$$\frac{d\,x(t)}{dt} + \frac{r_o\,x(t)}{V(t)} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 200 L of pure water. At $t = 0$, a salt water solution containing 0.2 kg salt per L is added at a rate of 15 L/min and the mixed solution is drained off at a rate of 10 L/min. How much salt is in the tank when it reaches the point of overflowing?

Solution: We have $r_i = 15, r_0 = 10, c_i = 0.2 = 1/5$, so $V = 200 + (r_i - r_0)t = 200 + 5t$. Hence the differential equation is

$$\frac{dx}{dt} + \frac{10\,x}{200+5t} = \frac{15}{5} \Leftrightarrow \frac{dx}{dt} + \frac{2\,x}{40+t} = 3.$$

Use the method of the integrating factor, with

$$\mu = e^{\int \frac{2dt}{40+t}} = e^{2\ln(40+t)} = (40+t)^2.$$

Then

$$x = \frac{1}{(40+t)^2}\int 3(40+t)^2\,dt = \frac{1}{(40+t)^2}\left((40+t)^3 + C\right) = 40 + t + \frac{C}{(40+t)^2}.$$

At $t = 0, x = 0$, so

$$0 = 40 + \frac{C}{40^2} \Rightarrow C = -40^3.$$

Finally, $V = 1000 \Leftrightarrow 200 + 5t = 1000 \Leftrightarrow t = 160$, and at that time:

$$x = 40 + 160 - \frac{40^3}{(40+160)^2} = 198.4,$$

exactly. So there will be 198.4 kg of salt in the tank when it reaches the point of overflowing.

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8. [10 marks; avg: 3.63/10] In the year 2000, a population of 25 finches was blown by a storm from the mainland to an offshore island that had no birds present at all. The finches quickly settled in and their population began to grow exponentially, so that their population doubled every two years. In the year 2010, another bad storm caused a large fallen tree from the mainland to wash up on the shore of the island. In the fallen tree was a small population of rats, who started feeding on the eggs in the finches' nests, causing the finch population to decrease at a rate proportional to the square of the finch population. If the population of finches was 1400 in the year 2012, what will be the population of finches on this island in the long run? (Use models we have covered in class.)

Solution: For the first 10 years the finch population grows exponentially, with doubling time 2 years. This means that, with t in years,

$$k = \frac{\ln 2}{2} \text{ and } P = 25\, e^{kt} = 25 \cdot 2^{t/2},$$

where P is the number of finches at time t. In particular, after 10 years the finch population is

$$P = 25 \cdot 2^{10/2} = 25 \cdot 2^5 = 25 \cdot 32 = 800.$$

After the rats arrive, the finch population grows logistically. That is, for k as before, and for some positive constant C,

$$\frac{dP}{dt} = kP - C\,P^2 \Leftrightarrow \frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right),$$

for some number L. Thus, remembering the solution, or by separating variables,

$$P = \frac{L}{1 + Be^{-kt}} = \frac{L}{1 + B \cdot 2^{-t/2}},$$

for some constant B. Now take $t = 0$ to be the year 2010. Then $t = 2$ corresponds to the year 2012; and we have two equations for B and L :

$$800 = \frac{L}{1+B} \Leftrightarrow 1 + B = \frac{L}{800}$$

and

$$1400 = \frac{L}{1 + B \cdot 2^{-1}} = \frac{2L}{2+B} \Leftrightarrow 2 + B = \frac{L}{700}.$$

We only need L. Subtract these two equations:

$$\frac{L}{700} - \frac{L}{800} = 1 \Leftrightarrow L\left(\frac{800 - 700}{700 \cdot 800}\right) = 1 \Leftrightarrow L = 5600.$$

(For interest, $B = 6$.) Since for logistic growth

$$\lim_{t \to \infty} P = L,$$

the population of the finches, in the long run, will be 5600.

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