MAT187H1F - Calculus II - Spring 2016

Solutions to Term Test 2 - June 15, 2016

Time allotted: 110 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

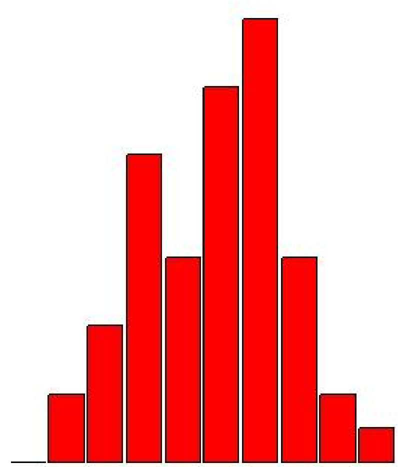
1. The results on this test were mediocre. Questions 1, 3, 4, 5 and 7 had passing averages, and Question 2 *nearly* had a passing average. So far so good. But Question 6 had 8 had very low averages—even though Question 6 was based almost verbatim on a Problem Set Question *and* a Suggested Exercise.

2. Question 8 was just the cosine version of examples done in class: Example 6 from Section 10.1 and Example 13 from Section 10.4. It shouldn't have been as mysterious as it seems to have appeared.

3. Be that as it may, I will count this test out of (top mark) 72, which increases the average to 58.5%. This produces a term mark of 34/50, approximately, or 68%.

Breakdown of Results: 54 students wrote this test. The marks ranged from 18.8% to 90%, and the average was 52.7%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	1.8%
A	5.5%	80-89%	3.7%
B	11.1%	70-79%	11.1%
C	24.1%	60-69%	24.1%
D	20.4%	50-59%	20.4%
F	38.9%	40-49%	11.1%
		30-39%	16.7%
		20-29%	7.4%
		10-19%	3.7%
		0-9%	0.0%



PART I : No explanation is necessary. Answer all of the following short-answer questions by putting your answer in the appropriate blank or by circling your choice(s).

1. [10 marks; 2 marks for each part. Avg: 6.7/10]

   (a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k =$ 2

$$\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{2}{3}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1} = \frac{2}{3}\left(\frac{1}{1-2/3}\right) = 2$$

   (b) $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2-1} =$ $\frac{3}{4}$

$$\sum_{k=2}^{\infty}\frac{1}{k^2-1} = \sum_{k=2}^{\infty}\frac{1}{2}\left(\frac{1}{k-1}-\frac{1}{k+1}\right) = \frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\cdots\right) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$$

   (c) The infinite series $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}$

   $\boxed{\text{(i)}}$ converges absolutely (*ii*) converges conditionally (*iii*) diverges

   (d) The infinite series $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{k+1}}$

   (*i*) converges absolutely $\boxed{\text{(ii)}}$ converges conditionally (*iii*) diverges

   (e) The infinite series $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\tan^{-1}x}$

   (*i*) converges absolutely (*ii*) converges conditionally $\boxed{\text{(iii)}}$ diverges

   Note: this was a misprint; it should have been $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\tan^{-1}k}$. But either way it diverges, and the 'uncorrected' version is actually easier.

2. [10 marks; 2 marks for each part. Avg: 4.0/10] Consider the power series $f(x) = \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{3}\right)^k$.

(a) What is the radius of convergence of $f(x)$? $R =$ 3

$$a_k = \frac{1}{3^k\,k};\ \lim_{k\to\infty}\frac{a_k}{a_{k+1}} = \lim_{k\to\infty}\frac{3^{k+1}(k+1)}{3^k k} = 3\lim_{k\to\infty}\frac{k+1}{k} = 3.$$

(b) What is the interval of convergence of $f(x)$? Answer: $[-3, 3)$

$$f(3) = \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{3}{3}\right)^k = \sum_{k=1}^{\infty}\frac{1}{k} \text{ diverges; } \ f(-3) = \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{-3}{3}\right)^k = \sum_{k=1}^{\infty}\frac{(-1)^k}{k} \text{ converges.}$$

(c) What is the power series for $f'(x)$, in sigma notation? $f'(x) = \sum_{k=1}^{\infty}\frac{1}{3}\left(\frac{x}{3}\right)^{k-1}$

(d) Find a formula for $f'(x)$ not in terms of a power series. $f'(x) = \frac{1}{3-x}$

$$\sum_{k=1}^{\infty}\frac{1}{3}\left(\frac{x}{3}\right)^{k-1} = \frac{1}{3}\left(\frac{1}{1-x/3}\right) = \frac{1}{3-x}$$

(e) What is the exact value of $f(2)$? $f(2) =$ $\ln 3$

$$f(x) = \int \frac{1}{3-x}\,dx = -\ln(3-x) + C;\ f(0) = 0 \Rightarrow C = \ln 3;$$
$$f(2) = -\ln 1 + \ln 3 = \ln 3$$

PART II : Present **complete** solutions to the following questions in the space provided.

3. [10 marks; avg: 6.6/10] Consider the curve with parametric equations $x = t^2 - 1,\ y = t^3 - 12t$.

(a) [5 marks] Find the first derivative $\dfrac{dy}{dx}$ and the coordinates of all the critical[1] points on the curve.

Solution:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 12}{2t} = \frac{3(t^2-4)}{2t} = \frac{3t}{2} - \frac{6}{t}, \text{ if } t \neq 0.$$

So $\dfrac{dy}{dx} = 0$ for $t = \pm 2$ and $\dfrac{dy}{dx}$ is undefined for $t = 0$. Thus the three critical points are

$$(x, y) = (-1, 0),\ (3, -16),\ (3, 16).$$

(b) [5 marks] Find the second derivative $\dfrac{d^2y}{dx^2}$ and the values of t for which the curve is concave up.

Solution: for $t \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\left(\frac{3}{2} + \frac{6}{t^2}\right)}{2t} = \frac{3(t^2+4)}{4t^3}.$$

Now find for which values of t the second derivative is positive:

$$\frac{d^2y}{dx^2} > 0 \Leftrightarrow t > 0.$$

[1] Recall: a critical point on a curve is a point where the derivative is zero or undefined.

4. [10 marks; avg: 5.8/10] Let $f(x) = e^{-x} \ln(1+x^2)$.

(a) [7 marks] Find the first four non-zero terms in the Maclaurin series of $f(x)$.

Solution: use

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots,$$
$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$$

and multiply them together, collecting like terms:

$$\begin{aligned} e^{-x}\ln(1+x^2) &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\right) \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \cdots - x^3 + \frac{x^5}{2} + \cdots + \frac{x^4}{2} - \frac{x^6}{4} + \cdots - \frac{x^5}{6} + \cdots \frac{x^6}{24} + \cdots \\ &= x^2 - x^3 + \frac{x^5}{3} + \frac{x^6}{8} + \cdots \end{aligned}$$

(b) [3 marks] What is the value of $f^{(5)}(0)$?

Solution:

$$\frac{1}{3} = \frac{f^{(5)}(0)}{5!} \Leftrightarrow f^{(5)}(0) = 40.$$

5. [10 marks; avg: 6.3/10] Approximate the value of

$$\int_0^{1/2} \frac{x}{\sqrt{1+x^6}}\,dx$$

correct to within 10^{-4}. Make sure to explain why your approximation *is* correct to within 10^{-4}.

Solution: use binomial series.

$$\begin{aligned}
\frac{x}{\sqrt{1+x^6}} &= x(1+x^6)^{-1/2} \\
&= x\left(1+(-1/2)x^6+\frac{(-1/2)(-3/2)}{2!}(x^6)^2+\frac{(-1/2)(-3/2)(-5/2)}{3!}(x^6)^3+\cdots\right) \\
&= x\left(1-\frac{x^6}{2}+\frac{3\,x^{12}}{8}-\frac{5\,x^{18}}{16}+\cdots\right) \\
&= x-\frac{x^7}{2}+\frac{3\,x^{13}}{8}-\frac{5\,x^{19}}{16}+\cdots \\
\Rightarrow \int_0^{1/2}\frac{x}{\sqrt{1+x^6}}\,dx &= \int_0^{1/2}\left(x-\frac{x^7}{2}+\frac{3\,x^{13}}{8}-\frac{5\,x^{19}}{16}+\cdots\right)dx \\
&= \left[\frac{x^2}{2}-\frac{x^8}{16}+\frac{3\,x^{14}}{112}-\frac{x^{20}}{64}+\cdots\right]_0^{1/2} \\
&= \frac{1}{8}-\frac{1}{4096}+\frac{3}{1835008}-\frac{1}{67108864}+\cdots \\
&= \underbrace{0.125-0.00024414}\cdots+0.000001634\cdots-0.000000014\cdots+\cdots \\
&= 0.124755859,\ \text{correct to within } 0.000001634<10^{-4},
\end{aligned}$$

by the alternating series test remainder formula.

6. [10 marks; avg: 3.7/10] Suppose a super ball is dropped from a height of 2 m and repeatedly bounces off the floor. Each time it hits the floor it rebounds to 75% of its previous height. Let the initial height of the ball be $h_0 = 2$; let h_n be the height of the ball after the n-th bounce, for $n \geq 1$. Let S_n be the total distance the ball has travelled at the time it hits the floor for the n-th time.

(a) [4 marks] Write down formulas for h_n and S_n.

Solution: $h_n = 2\left(\frac{3}{4}\right)^n$;

$$\begin{aligned} S_n &= h_0 + 2h_1 + 2h_2 + \cdots 2h_{n-1} \\ &= \sum_{k=0}^{n-1} 2h_k - h_0 \\ &= 4\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k - 2 \\ &= 4\left(\frac{1-(3/4)^n}{1-3/4}\right) - 2 \\ &= 14 - 16\left(\frac{3}{4}\right)^n \end{aligned}$$

(b) [3 marks] What is the total distance travelled by the ball when it finally stops bouncing?

Solution: depending on how far you simplified your answer for S_n in part (a), this could be a short calculation,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(14 - 16\left(\frac{3}{4}\right)^n\right) = 14,$$

or a longer calculation

$$\lim_{n\to\infty} S_n = 4\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k - 2 = 4\left(\frac{1}{1-3/4}\right) - 2 = 16 - 2 = 14.$$

(c) [3 marks] What is the total time taken before the ball comes to rest? Assume $g = 9.8$ m/sec^2.

Solution: use $h_n = \frac{1}{2}gt_n^2 \Rightarrow t_n = \sqrt{\frac{2h_n}{g}}$. Then the total time taken is

$$2\sum_{k=0}^{\infty} t_k - t_0 = 2\sum_{k=0}^{\infty}\sqrt{\frac{4}{g}}\left(\sqrt{\frac{3}{4}}\right)^k - \sqrt{\frac{4}{g}} = \sqrt{\frac{4}{g}}\left(\frac{2}{1-\sqrt{3/4}} - 1\right) = \sqrt{\frac{4}{g}}\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) \approx 8.9 \text{ sec}$$

7. [10 marks; avg: 6.4/10] For the given convergent infinite series determine *at least* how many terms of the series must be added up to approximate the sum of the series correctly to within 10^{-4}.

   (a) [4 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$.

   Solution: use the alternating series remainder term.

$$|R_n| < a_{n+1} = \frac{1}{(n+1)^4} < 10^{-4} \Rightarrow (n+1)^4 > 10^4 \Rightarrow n+1 > 10 \Rightarrow n > 9.$$

   So you have to add at least 10 terms. (Actually, 8 will do, but this is not easy to show without a computer algebra system.)

   (b) [6 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$.

   Solution: use the integral test remainder term.

$$R_n < \int_n^{\infty} \frac{1}{x^4}\,dx = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_n^b = \frac{1}{3n^3} < 10^{-4} \Rightarrow \frac{10000}{3} < n^3 \Rightarrow n > 10\left(\frac{10}{3}\right)^{1/3} \approx 14.9380.$$

   So you have to add at least 15 terms.

8. [10 marks; avg: 2.7/10] Consider the problem:

> For which values of x does the polynomial $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ approximate $\cos x$ correctly to within 10^{-6}?

Solve this problem in two ways: one using Taylor's remainder formula, one without using Taylor's remainder formula.

Solution: no matter which way you do this, you have to realize that $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ is a Maclaurin polynomial approximation to $f(x) = \cos x$.

With Taylor's remainder formula: observe that since $f^{(5)}(0) = -\sin 0 = 0$,

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} = P_4(x) = P_5(x).$$

Then

$$R_5(x) = \frac{f^{(6)}(z)}{6!}x^6,$$

for some number z between 0 and x. Since $f^{(6)}(z) = -\cos z$, we have

$$|R_5(x)| = \left|\frac{-\cos(z)}{6!}x^6\right| = |\cos z|\frac{|x|^6}{6!} \leq 1 \cdot \frac{|x|^6}{6!} = \frac{|x|^6}{6!}.$$

Take x such that

$$|x|^6 < 6! \cdot 10^{-6} \Leftrightarrow |x| < \frac{720^{1/6}}{10} \approx 0.299379516\ldots$$

Without Taylor's remainder formula: use the Maclaruin series for $\cos x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = P_4(x) - \frac{x^6}{6!} + \cdots$$

By the alternating series test remainder form, if you approximate $\cos x$ by $P_4(x)$, then the error term R_4 satisfies

$$|R_4| < \frac{x^6}{6!}.$$

Now set

$$\frac{x^6}{6!} < 10^{-6} \Leftrightarrow |x| < \frac{720^{1/6}}{10} \approx 0.299379516\ldots,$$

as before. (Aside: is

$$a_k = \frac{x^{2k}}{(2k)!}$$

a decreasing sequence? Certainly if $|x| < 1$, which is good enough for this problem.)

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