Solutions to MAT188H1S Quiz 4

1. (5 marks) Define following:

   (a) [2 marks] the non-empty set of vectors S is a subspace of $\mathbb{R}^n$.

   Definition: the non-empty set of vectors S is a subspace of $\mathbb{R}^n$ if

   1. $\vec{u}, \vec{v} \in S \Rightarrow \vec{u} + \vec{v} \in S$. (S is closed under vector addition.)
   2. $a \in \mathbb{R}, \vec{v} \in S \Rightarrow a\,\vec{v} \in S$. (S is closed under scalar multiplication.)

   (b) [2 marks] the set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ is linearly independent.

   Definition: the set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ is linearly independent if

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_m\vec{u}_m = \vec{0} \Rightarrow a_1 = 0, a_2 = 0, \ldots, \vec{a}_m = 0,$$

   where $a_1, a_2, \ldots, a_m$ are scalars in $\mathbb{R}$. More formally,

$$\sum_{i=1}^{m} a_i\vec{u}_i = \vec{0} \Rightarrow a_i = 0.$$

   In words:

   The only linear combination of the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ that is equal to the zero vector is the trivial linear combination.

   (c) [1 mark] span$\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$.

   Definition:

$$\text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\} = \left\{ \sum_{i=1}^{m} a_i\vec{u}_i \mid a_i \in \mathbb{R}, 1 \leq i \leq m \right\}.$$

   In words:

   span$\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ is the set of all possible linear combinations of the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$.

2. (5 marks) Find a basis for null(A) if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

Solution: find the basic solution(s) to the homogeneous system of equations $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 4 & 6 & 7 & 8 & 0 \\ 9 & 10 & 11 & 12 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -2 & -5 & -8 & 0 \\ 0 & -8 & -16 & -24 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & 5 & 8 & 0 \\ 0 & 0 & 4 & 8 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & 5 & 8 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

Let $x_4 = t$ be a parameter. The the general solution to $Ax = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

So a basis for the null space of A is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

3. (5 marks) Find a basis for the subspace span $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

Solution 1: by inspection. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Then $\vec{v}_2 \neq k\vec{v}_1$, but $v_3 = 2\vec{v}_1 - \vec{v}_2$. So a basis for $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is

$$\{\vec{v}_1, \vec{v}_2\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} \right\}.$$

Solution 2: reduce the matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ to find its independent columns.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 2 \\ -1 & -4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

So a basis for $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the set $\{\vec{v}_1, \vec{v}_2\}$, consisting of the columns of A that correspond to the columns of R with the leading 1's.