University of Toronto

Faculty of Applied Science and Engineering

Solutions to Final Examination, April 2017

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

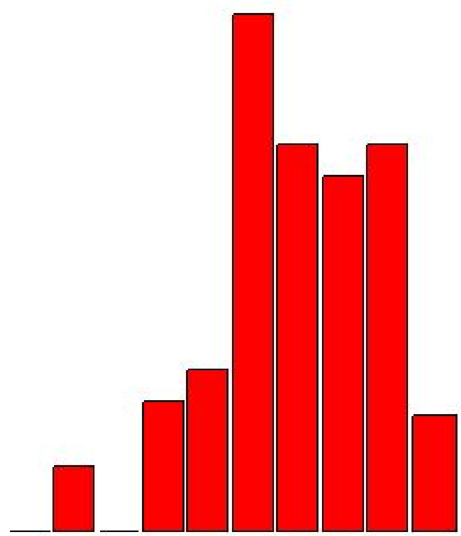
MAT188H1S - Linear Algebra

Examiner: D. Burbulla

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Breakdown of Results: 66 students wrote this exam. The marks ranged from 17.7% to 96.7%, and the average was 64.6%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	6.1%
A	24.3%	80-89%	18.2%
B	16.7%	70-79%	16.7%
C	18.2%	60-69%	18.2%
D	24.2%	50-59%	24.2%
F	16.7%	40-49%	7.6 %
		30-39%	6.1%
		20-29%	0.0%
		10-19%	3.0%
		0-9%	0.0%



1. [avg: 8/10] Given that the reduced row echelon form of

$$A = \begin{bmatrix} 2 & 10 & 1 & 7 & 5 \\ -1 & -5 & 1 & 1 & -7 \\ 2 & 10 & 1 & 7 & 5 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 5 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. (No justification is required.)

(a) [1 mark] the rank of A Answer: 2

(b) [1 mark] $\dim(\text{Row}(A))$ Answer: 2

(c) [1 mark] $\dim(\text{Col}(A))$ Answer: 2

(d) [1 mark] $\dim(\text{Null}(A))$ Answer: 3

(e) [1 mark] $\dim(\text{Null}(A^T))$ Answer: 1

(f) [1 mark] A basis for the row space of A.

Solution: any two independent rows of A or R will do: Answer: $\begin{bmatrix} 1 \\ 5 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ -3 \end{bmatrix}$

(g) [2 marks] A basis for the column space of A.

Solution: any two independent columns of A will do: Answer: $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(h) [2 marks] A basis for the null space of A.

Solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5s - 2t - 4u \\ s \\ -3t + 3u \\ t \\ u \end{bmatrix}$ Answer: $\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

2. [avg: 8.7/10]

Let

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ 10 \\ 3 \\ -11 \end{bmatrix}.$$

Show $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthogonal set, and write $\vec{x}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$.

Solution: need to show $\vec{u}_i \cdot \vec{u}_j = 0$, for $i \neq j$. Since $\vec{u}_i \cdot \vec{u}_j = \vec{u}_j \cdot \vec{u}_i$, there are only six dot products you must check:

$$\vec{u}_1 \cdot \vec{u}_2 = -2 - 1 + 3 + 0 = 0, \quad \vec{u}_1 \cdot \vec{u}_3 = 1 - 1 + 0 + 0 = 0, \quad \vec{u}_1 \cdot \vec{u}_4 = 1 - 10 + 9 + 0 = 0,$$

$$\vec{u}_2 \cdot \vec{u}_3 = -2 + 1 + 0 + 1 = 0, \quad \vec{u}_2 \cdot \vec{u}_4 = -2 + 10 + 3 - 11 = 0, \quad \vec{u}_3 \cdot \vec{u}_4 = 1 + 10 + 0 - 11 = 0.$$

Thus $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an independent set of four vectors in $\mathbb{R}^4$, so it is a basis for $\mathbb{R}^4$, and

$$\begin{aligned} \vec{x} &= \frac{\vec{x} \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 + \frac{\vec{x} \cdot \vec{u}_2}{\|\vec{u}_2\|^2}\vec{u}_2 + \frac{\vec{x} \cdot \vec{u}_3}{\|\vec{u}_3\|^2}\vec{u}_3 + \frac{\vec{x} \cdot \vec{u}_4}{\|\vec{u}_4\|^2}\vec{u}_4 \\ &= \frac{1+1+3}{1+1+9}\vec{u}_1 + \frac{-2-1+1-1}{4+1+1+1}\vec{u}_2 + \frac{1-1-1}{1+1+1}\vec{u}_3 + \frac{1-10+3+11}{1+100+9+121}\vec{u}_4 \\ &= \frac{5}{11}\vec{u}_1 - \frac{3}{7}\vec{u}_2 - \frac{1}{3}\vec{u}_3 + \frac{5}{231}\vec{u}_4 \end{aligned}$$

Alternate Solution: solve the vector equation $\vec{x} = x_1\,\vec{u}_1 + x_2\,\vec{u}_2 + x_3\,\vec{u}_3 + x_4\,\vec{u}_4$ for x_1, x_2, x_3, x_4 by solving a linear system of equations. Use augmented matrices and good old row reduction:

$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 1 \\ -1 & 1 & 1 & 10 & -1 \\ 3 & 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & -11 & -1 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 1 \\ 0 & -1 & 2 & 11 & 0 \\ 0 & 4 & 3 & 33 & -2 \\ 0 & 1 & 1 & -11 & -1 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 1 \\ 0 & 1 & -2 & -11 & 0 \\ 0 & 0 & 11 & 77 & -2 \\ 0 & 0 & 3 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 1 \\ 0 & 1 & -2 & -11 & 0 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 7 & 5/33 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 5/11 \\ 0 & 1 & 0 & 0 & -3/7 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & 5/231 \end{array}\right],$$

from which

$$x_1 = \frac{5}{11},\; x_2 = -\frac{3}{7},\; x_3 = -\frac{1}{3},\; x_4 = \frac{5}{231},$$

as before.

3. [avg: 6.2/10]

Let $A = \dfrac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$. Find the eigenvalues of A and a basis for each eigenspace of A. Plot the eigenspaces of A in $\mathbb{R}^2$, and clearly indicate which eigenspace corresponds to which eigenvalue. Interpret your result geometrically.

Solution: be careful with the fractions!

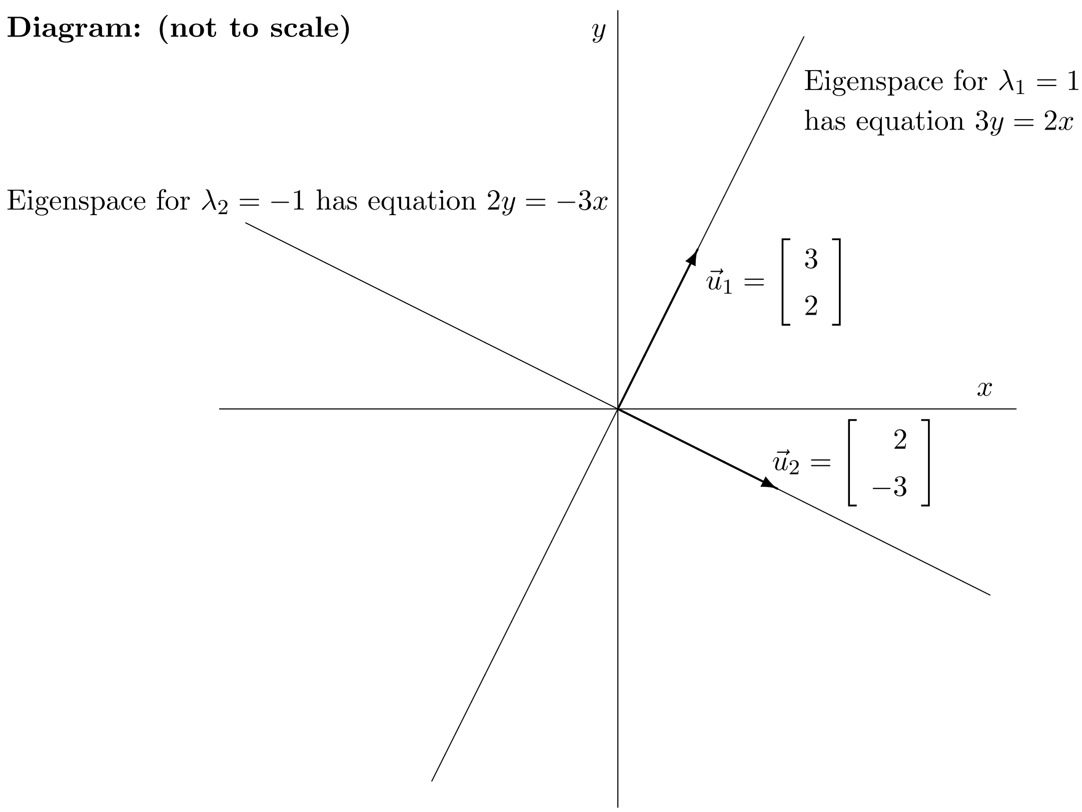
$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda - 5/13 & -12/13 \\ -12/13 & \lambda + 5/13 \end{bmatrix} = \lambda^2 - \frac{25}{169} - \frac{144}{169} = \lambda^2 - 1,$$

so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. Next, find a basis for each eigenspace:

$$\text{null}\begin{bmatrix} \lambda_1 - 5/13 & -12/13 \\ -12/13 & \lambda_1 + 5/13 \end{bmatrix} = \text{null}\begin{bmatrix} 8/13 & -12/13 \\ -12/13 & 18/13 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\};$$

$$\text{null}\begin{bmatrix} \lambda_2 - 5/13 & -12/13 \\ -12/13 & \lambda_2 + 5/13 \end{bmatrix} = \text{null}\begin{bmatrix} -18/13 & -12/13 \\ -12/13 & -8/13 \end{bmatrix} = \text{null}\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right\}.$$

Diagram: (not to scale)



The eigenvectors are orthogonal to each other; each eigenspace is a line passing through the origin.

Geometrical Interpretation: A is the matrix of a reflection in the line with equation $3y = 2x$.

4. [avg: 7.4/10] Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 \\ 9x_1 + 2x_2 \end{bmatrix}.$$

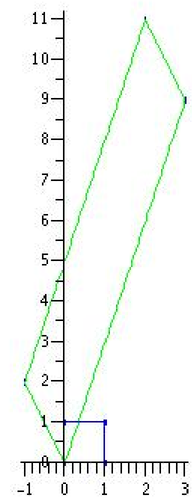
(a) [5 marks] Draw the image of the unit square[1] under L and find its area.

Solution:

$$[L] = \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 9 & 2 \end{bmatrix}$$

The image of the image of the unit square is to the right; the area of this parallelogram is

$$\left|\det \begin{bmatrix} 3 & -1 \\ 9 & 2 \end{bmatrix}\right| = 15$$



(b) [5 marks] Find $L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

Solution: use the formula for the inverse of a 2×2 matrix.

$$[L^{-1}] = [L]^{-1} = \begin{bmatrix} 3 & -1 \\ 9 & 2 \end{bmatrix}^{-1} = \frac{1}{6+9}\begin{bmatrix} 2 & 1 \\ -9 & 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 2 & 1 \\ -9 & 3 \end{bmatrix}.$$

Thus

$$L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{15}\begin{bmatrix} 2 & 1 \\ -9 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 2x_1 + x_2 \\ -9x_1 + 3x_2 \end{bmatrix}.$$

[1]The unit square is the square with the four vertices $(0,0), (1,0), (0,1), (1,1)$.

5. [avg: 2.4/10]

Let

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 6 \\ 3 & 2 & 7 & 1 \\ 3 & 6 & 7 & 4 \end{bmatrix}.$$

(a) [5 marks] Show that $S = \{\vec{x} \in \mathbb{R}^4 \mid A\,\vec{x} = A^T\,\vec{x}\}$ is a subspace of $\mathbb{R}^4$.

Solution: the easy way is to observe that $S = \text{null}(A - A^T)$, so S is a subspace.

5.(b) [5 marks] Find a basis for S.

Solution:

$$\begin{aligned} \text{null}(A - A^T) &= \text{null} \begin{bmatrix} 2-2 & 1-1 & 3-3 & 4-3 \\ 1-1 & 5-5 & 2-2 & 6-6 \\ 3-3 & 2-2 & 7-7 & 1-7 \\ 3-4 & 6-6 & 7-1 & 4-4 \end{bmatrix} \\ &= \text{null} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \\ -1 & 0 & 6 & 0 \end{bmatrix} \\ &= \text{null} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Note that the rank of $A - A^T$ is 2, so that $\dim(\text{null}(A - A^T)) = 4 - 2$. A basis for S is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

6. [avg: 6.1/10] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 7 & 1 & -1 \\ 1 & 7 & -1 \\ -1 & -1 & 7 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\begin{aligned}
\det\begin{bmatrix} \lambda-7 & -1 & 1 \\ -1 & \lambda-7 & 1 \\ 1 & 1 & \lambda-7 \end{bmatrix} &= \det\begin{bmatrix} \lambda-7 & -1 & 1 \\ -1 & \lambda-7 & 1 \\ 0 & \lambda-6 & \lambda-6 \end{bmatrix} = (\lambda-6)\det\begin{bmatrix} \lambda-7 & -1 & 1 \\ -1 & \lambda-7 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
&= (\lambda-6)\det\begin{bmatrix} \lambda-7 & -2 & 1 \\ -1 & \lambda-8 & 1 \\ 0 & 0 & 1 \end{bmatrix} = (\lambda-6)\det\begin{bmatrix} \lambda-7 & -2 \\ -1 & \lambda-8 \end{bmatrix} \\
&= (\lambda-6)(\lambda^2-15\lambda+54) = (\lambda-6)(\lambda-6)(\lambda-9) = (\lambda-6)^2(\lambda-9)
\end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 6$, repeated, and $\lambda_2 = 9$.

Step 2: recall null$(A - \lambda I)$ is the eigenspace of A corresponding to the eigenvalue λ.
Find three mutually **orthogonal** eigenvectors of A. For $\lambda_2 = 9$, null$(9\,I - A) =$

$$\text{null}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 0 & 3 & 3 \\ -1 & 2 & 1 \\ 0 & 3 & 3 \end{bmatrix} = \text{null}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\},$$

and for $\lambda_1 = 6$, null$(6\,I - A) =$

$$\text{null}\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}.$$

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}; \quad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

7. [avg: 6.4/10] Let $S = \text{span}\left\{\begin{bmatrix} 1 & 0 & -1 & 1 \end{bmatrix}^T, \begin{bmatrix} 3 & 1 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 2 & 0 & 1 & 1 \end{bmatrix}^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

Solution: call the three given vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ and apply the Gram-Schmidt algorithm to find an orthogonal basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Take $\vec{v}_1 = \vec{x}_1$,

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix},$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis of S.

(b) [5 marks] Let $\vec{x} = \begin{bmatrix} 2 & -3 & 3 & 4 \end{bmatrix}^T$. Find $\text{proj}_S(\vec{x})$.

Solution: use the projection formula, and your orthogonal basis from part (a).

$$\text{proj}_S\vec{x} = \frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 + \frac{\vec{x} \cdot \vec{v}_3}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \frac{7}{9}\begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} + \frac{13}{6}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 46 \\ -64 \\ 49 \\ 57 \end{bmatrix}.$$

Alternate Solution: observe that $S^\perp = \text{span}\{\vec{y}\}$ with $\vec{y} = \begin{bmatrix} -2 & 2 & 1 & 3 \end{bmatrix}^T$, since

$$\text{null}\begin{bmatrix} 1 & 0 & -1 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 3 & -1 \end{bmatrix} = \text{null}\begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 3 & -1 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -2 \\ 2 \\ 1 \\ 3 \end{bmatrix}\right\}$$

Then

$$\text{proj}_S(\vec{x}) = \vec{x} - \text{proj}_{S^\perp}(\vec{x}) = \vec{x} - \frac{\vec{x} \cdot \vec{y}}{\|\vec{y}\|^2}\vec{y} = \vec{x} - \frac{5}{18}\vec{y} = \frac{1}{18}\begin{bmatrix} 46 & -64 & 49 & 57 \end{bmatrix}^T.$$

8. [avg: 7.2/10] Consider the plane Π in $\mathbb{R}^3$ with scalar equation $2x_1 - 3x_2 - 5x_3 = 0$ and the point with coordinates $Q(1, 3, 1)$. Find the point on the plane Π closest to the point Q.

Solution 1: let R be the point on the plane closest to Q. Then, by the Best Approximation Theorem, $\overrightarrow{OR} = \text{proj}_{\Pi}(\overrightarrow{OQ})$. To use this formula, though, we need an orthogonal basis of Π : take (by trial and error)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 8 \\ 7 \\ -1 \end{bmatrix}.$$

(Or, take any vector in Π and cross it with the normal vector to Π; that will give you another vector in Π; or find a basis of Π and then apply the Gram-Schmidt Algorithm to it.) Then

$$\begin{aligned} \overrightarrow{OR} = \text{proj}_{\Pi}(\overrightarrow{OQ}) &= \frac{\overrightarrow{OQ} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\, \vec{v}_1 + \frac{\overrightarrow{OQ} \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\, \vec{v}_2 \\ &= -\frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{14}{57} \begin{bmatrix} 8 \\ 7 \\ -1 \end{bmatrix} \\ &= \frac{1}{19} \begin{bmatrix} 31 \\ 39 \\ -11 \end{bmatrix} \end{aligned}$$

Solution 2: let $\mathcal{L}$ be the line that passes through the point Q and is perpendicular to the plane Π. Then $\mathcal{L}$ has parametric equations

$$\begin{cases} x_1 &= 1 + 2t \\ x_2 &= 3 - 3t \\ x_3 &= 1 - 5t \end{cases}$$

The closest point on Π to the point Q is the point R, which is the intersection of the line $\mathcal{L}$ and the plane Π. To find this intersection point substitute from the line into the plane, and solve for t:

$$2(1 + 2t) - 3(3 - 3t) - 5(1 - 5t) = 0 \Leftrightarrow 38t - 12 = 0 \Leftrightarrow t = \frac{6}{19}.$$

Then R has coordinates

$$\left(1 + 2\left(\frac{6}{19}\right), 3 - 3\left(\frac{6}{19}\right), 1 - 5\left(\frac{6}{19}\right)\right) = \left(\frac{31}{19}, \frac{39}{19}, -\frac{11}{19}\right)$$

9. [avg: 5.8/10] Use the method of least squares to find the best fitting quadratic function for the five data points

$$(-2, 4.9),\ (-1, 2.1),\ (0, 0.9),\ (1, 1.9),\ (2, 5.1).$$

Solution: take $x_1 = -2,\ x_2 = -1,\ x_3 = 0,\ x_4 = 1,\ x_5 = 2$ and $\vec{y} = \begin{bmatrix} 4.9 \\ 2.1 \\ 0.9 \\ 1.9 \\ 5.1 \end{bmatrix}$. Let the quadratic function be $y = a + b\,x + c\,x^2$. We have

$$M = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \text{ and } M^T M = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}.$$

The normal equations are

$$M^T M \begin{bmatrix} a \\ b \\ c \end{bmatrix} = M^T \vec{y} \Leftrightarrow \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 14.9 \\ 0.2 \\ 44.0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 14.9 \\ 0.2 \\ 14.2 \end{bmatrix}$$

This system is actually easy to solve directly, since b is already completely isolated:

$$a = 333/350, b = 1/50, c = 71/70.$$

OR: Use the Gaussian algorithm to find the inverse of $M^T M$:

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 10 & 1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 1 & 0 \\ 10 & 0 & 34 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/10 & 0 \\ 0 & 0 & 14 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 7 & 0 & 0 & 17/5 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/10 & 0 \\ 0 & 0 & 14 & -2 & 0 & 1 \end{array}\right]$$

So

$$(M^T M)^{-1} = \begin{bmatrix} 17/35 & 0 & -1/7 \\ 0 & 1/10 & 0 \\ -1/7 & 0 & 1/14 \end{bmatrix}; \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 17/35 & 0 & -1/7 \\ 0 & 1/10 & 0 \\ -1/7 & 0 & 1/14 \end{bmatrix} \begin{bmatrix} 14.9 \\ 0.2 \\ 44.0 \end{bmatrix} = \begin{bmatrix} 33.3/35 \\ 0.02 \\ 35.5/35 \end{bmatrix}.$$

Either way, the best fitting quadratic to the data is

$$y = \frac{333}{350} + \frac{1}{50}\,x + \frac{71}{70}\,x^2 \approx 0.951428571 + 0.02\,x + 1.014285714\,x^2.$$

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