University of Toronto
Faculty of Applied Science and Engineering
Solutions to Final Examination, April 2019
Duration: 2 and 1/2 hrs
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS
MAT188H1S - Linear Algebra
Examiner: D. Burbulla

Exam Type: A.

Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

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1. [10 marks] Given that the reduced row echelon form of

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & -2 \\ 3 & 5 & 8 & 2 & 4 \\ 6 & 11 & 17 & 5 & -2 \\ -2 & 4 & 2 & 6 & -5 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the rank of A and a basis for each of $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$.

Solution:

- by definition, the rank of A is 3, the number of leading 1's in R.
- a basis for $\text{row}(A)$ consists of the three non-zero rows of R,

$$\{\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}\},$$

  OR any three *independent* rows of A, which you must demonstrate *are* independent.
- a basis for $\text{col}(A)$ consists of the columns of A that correspond to the columns of R with leading 1's,

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 11 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \\ -5 \end{bmatrix} \right\},$$

  OR any other three *independent* columns of A, which you must demonstrate *are* independent.
- a basis for $\text{null}(A)$ consists of the basic solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, which can be read off R :

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

  Aside: if not by inspection then you can get the basic solutions to $A\vec{x} = \vec{0}$ by finding the general solution and writing it as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s+t \\ -s-t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

  where s and t are parameters.

2. [10 marks] For this question, let A be a 4×7 matrix.

(a) [5 marks] Find all possible values of $\dim(\operatorname{im}(A))$ and the corresponding values of $\dim(\operatorname{null}(A))$.

Solution: for this whole question let r be the rank of A. We have

$$\dim(\operatorname{null}(A)) = 7 - r \text{ and } \dim(\operatorname{im}(A)) = \dim(\operatorname{col}(A)) = \dim(\operatorname{row}(A)) = r.$$

Since A can have at most 4 independent rows, we know $r \leq 4$. Thus the five possibilities are

$$(\dim(\operatorname{im}(A)), \dim(\operatorname{null}(A))) = (0,7), (1,6), (2,5), (3,4) \text{ or } (4,3).$$

(b) [5 marks; 1 mark for each part] Now suppose the rank of A is 4. Find the value of the following:

(i) $\dim(\operatorname{col}(A))$

Solution: $\dim(\operatorname{col}(A)) = r = 4$

(ii) $\dim(\operatorname{row}(A))$

Solution: $\dim(\operatorname{row}(A)) = r = 4$

(iii) $\dim\left((\operatorname{null}(A))^\perp\right)$

Solution: $\dim\left((\operatorname{null}(A))^\perp\right) = 7 - \dim(\operatorname{null}(A)) = 7 - 3 = 4$

(iv) $\dim\left(\operatorname{null}(A^T)\right)$

Solution: since A^T is 7×4 and A and A^T have the same rank, $\dim\left(\operatorname{null}(A^T)\right) = 4 - 4 = 0$

(v) $\dim\left(\operatorname{null}(AA^T)\right)$

Solution: by a Theorem in the book, AA^T is invertible, so $\dim\left(\operatorname{null}(AA^T)\right) = 0$

3. [14 marks] If A is an $m \times n$ matrix it can be proved that there is a unique $n \times m$ matrix A^* (called the *generalized inverse* of A) satisfying the following four conditions:

$$\underbrace{A\,A^*\,A = A}_{(1)},\ \underbrace{A^*\,A\,A^* = A^*}_{(2)},\ \underbrace{(AA^*)^T = AA^*}_{(3)} \text{ and } \underbrace{(A^*A)^T = A^*A}_{(4)}.$$

(a) [6 marks] Verify that the generalized inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is $A^* = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}$.

Solution: show that A^* and A satisfy all (4) conditions.

(3) $AA^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is symmetric.

(4) $A^*A = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ is also symmetric.

(1) $AA^*A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A$

(2) $A^*AA^* = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} = A^*$

(b) [2 marks] Show that if A is an $n \times n$ invertible matrix, then $A^* = A^{-1}$.

Solution: (3) and (4) check out since $AA^{-1} = I = A^{-1}A$, and I is symmetric.
(1) checks out since $AA^{-1}A = IA = A$, and (2) checks out too, since $A^{-1}AA^{-1} = IA^{-1} = A^{-1}$.

(c) [2 marks] Show that if A is an $n \times n$ symmetric matrix such that $A^2 = A$, then $A^* = A$.

Solution: (3) and (4) check out since $AA = A^2 = A$, and A is given as symmetric.
(1) and (2) check out since $AAA = A^2A = AA = A^2 = A$.

(d) [4 marks] Show that if $\vec{b} \in \mathbb{R}^m$ and $A\,A^*\vec{b} = \vec{b}$, then for any vector $\vec{y} \in \mathbb{R}^n$, $\vec{x} = A^*\vec{b} + (I - A^*A)\,\vec{y}$ is a solution to the matrix equation $A\,\vec{x} = \vec{b}$.

Solution: check that the formula for $\vec{x}$ satisfies the equation $A\,\vec{x} = \vec{b}$:

$$\begin{aligned} A(A^*\vec{b} + (I - A^*A)\,\vec{y}) &= AA^*\vec{b} + A(I - A^*A)\vec{y} \\ &= \vec{b} + (A - AA^*A)\vec{y}, \text{ since it is given that } A\,A^*\vec{b} = \vec{b} \\ &= \vec{b} + (A - A)\vec{y}, \text{ by property (1) of } A^* \\ &= \vec{b} + \vec{0} \\ &= \vec{b} \end{aligned}$$

4. [6 marks] Find the least squares approximating line $y = z_0 + z_1\,x$ for the data points $(2,4), (4,3), (7,2), (8,1)$.

Solution: use the normal equations. Let

$$M = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 7 \\ 1 & 8 \end{bmatrix},\ M^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 7 & 8 \end{bmatrix},\ Z = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix},\ Y = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

Solve the normal equations for Z :

$$M^TMZ = M^TY \Leftrightarrow \begin{bmatrix} 4 & 21 \\ 21 & 133 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 10 \\ 42 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 4 & 21 \\ 21 & 133 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 42 \end{bmatrix} = \frac{1}{91} \begin{bmatrix} 133 & -21 \\ -21 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 42 \end{bmatrix} = \frac{1}{91} \begin{bmatrix} 448 \\ -42 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 64 \\ -6 \end{bmatrix}$$

So the least squares approximating line to the data has equation

$$y = \frac{64}{13} - \frac{6\,x}{13}.$$

5. [10 marks] Prove the following:

(a) [3 marks] If A and B are $m \times n$ matrices and $U = \{\vec{x} \in \mathbb{R}^n \mid A\,\vec{x} = B\,\vec{x}\}$, then U is a subspace of $\mathbb{R}^n$.

Proof: the easiest way is to observe that $U = \text{null}(A - B)$.

(b) [4 marks] Eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

Proof: suppose A is a symmetric matrix, λ_1, λ_2 are two distinct eigenvalues of A, and $\vec{x}_1, \vec{x}_2$ are corresponding eigenvectors of A. Then $A^T = A$ and

$$\lambda_1(\vec{x}_1 \cdot \vec{x}_2) = (\lambda_1\vec{x}_1) \cdot \vec{x}_2 = (A\,\vec{x}_1) \cdot \vec{x}_2 = (A\,\vec{x}_1)^T\vec{x}_2 = (\vec{x}_1^T A^T)\,\vec{x}_2$$

$$= \vec{x}_1^T(A^T\vec{x}_2) = \vec{x}_1^T(A\,\vec{x}_2) = \vec{x}_1^T(\lambda_2\vec{x}_2) = \vec{x}_1 \cdot (\lambda_2\,\vec{x}_2) = \lambda_2(\vec{x}_1 \cdot \vec{x}_2)$$

$$\Rightarrow (\lambda_1 - \lambda_2)\,\vec{x}_1 \cdot \vec{x}_2 = 0 \Rightarrow \vec{x}_1 \cdot \vec{x}_2 = 0, \text{ since } \lambda_1 \neq \lambda_2.$$

(c) [3 marks] $\lambda = 0$ is an eigenvalue of the $n \times n$ matrix A if and only if A is not invertible.

Solution:

$$\begin{aligned} \lambda = 0 \text{ is an eigenvalue of } A \quad &\Leftrightarrow \quad \det(A - 0\,I) = 0 \\ &\Leftrightarrow \quad \det(A) = 0 \\ &\Leftrightarrow \quad A \text{ is not invertible} \end{aligned}$$

6. [10 marks] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}.$$

Step 1: find the eigenvalues of A.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 8 & 2 & -2 \\ 2 & \lambda - 5 & -4 \\ -2 & -4 & \lambda - 5 \end{bmatrix} = \det \begin{bmatrix} \lambda - 8 & 2 & 0 \\ 2 & \lambda - 5 & \lambda - 9 \\ -2 & -4 & \lambda - 9 \end{bmatrix}$$

$$= (\lambda - 9) \det \begin{bmatrix} \lambda - 8 & 2 & 0 \\ 2 & \lambda - 5 & 1 \\ -2 & -4 & 1 \end{bmatrix} = (\lambda - 9) \det \begin{bmatrix} \lambda - 8 & 2 & 0 \\ 4 & \lambda - 1 & 0 \\ -2 & -4 & 1 \end{bmatrix} = (\lambda - 9)(\lambda^2 - 9\lambda) = \lambda(\lambda - 9)^2$$

Thus the eigenvalues of A are $\lambda_1 = 9$, repeated, and $\lambda_2 = 0$.

Step 2: find an *orthogonal* basis of eigenvectors for each eigenspace.

$$E_9(A) = \text{null} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\};$$

$$E_0(A) = \text{null} \begin{bmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & -18 & -18 \\ 2 & -5 & -4 \\ 0 & -9 & -9 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

Step 3: for the columns of P, take the unit, orthogonal eigenvectors and for the diagonal entries of D take the corresponding eigenvalues:

$$P = \begin{bmatrix} 0 & 4/\sqrt{18} & 1/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \end{bmatrix} \text{ and } D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

7. [10 marks] Let $U = \text{span}\left\{\underbrace{[\ 1\ \ 1\ \ 1\ \ 0\ ]^T}_{\vec{x}_1}, \underbrace{[\ 1\ \ 0\ \ 1\ \ 1\ ]^T}_{\vec{x}_2}, \underbrace{[\ 1\ \ 0\ \ 0\ \ 1\ ]^T}_{\vec{x}_3}\right\}$.

(a) [5 marks] Find an orthogonal basis of U.

Solution: use the Gram-Schmidt algorithm. Call the three given vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$, respectively. Take $\vec{f}_1 = \vec{x}_1$. Then

$$\vec{f}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix},$$

$$\vec{f}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 - \frac{\vec{x}_3 \cdot \vec{f}_2}{\|\vec{f}_2\|^2}\vec{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{15}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 6 \\ 3 \\ -9 \\ 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix}.$$

(b) [5 marks] Let $\vec{x} = \begin{bmatrix} -2 & 1 & 0 & 3 \end{bmatrix}^T$. Find $\text{proj}_U(\vec{x})$.

Solution: use the projection formula, for which you must use an orthogonal basis of U :

$$\begin{aligned} \text{proj}_U(\vec{x}) &= \frac{\vec{x} \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 + \frac{\vec{x} \cdot \vec{f}_2}{\|\vec{f}_2\|^2}\vec{f}_2 + \frac{\vec{x} \cdot \vec{f}_3}{\|\vec{f}_3\|^2}\vec{f}_3 \\ &= -\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{15}\begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix} + \frac{0}{15}\begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Alternate solution: $U^\perp = \text{null}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Then $\text{proj}_U(\vec{x}) = \vec{x} - \text{proj}_{U^T}(\vec{x}) = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix} - \frac{6}{3}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, as before.

8. [10 marks] Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation with matrix $A = \begin{bmatrix} \vec{e}_n & \vec{e}_{n-1} & \dots & \vec{e}_2 & \vec{e}_1 \end{bmatrix}$, where $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}, \vec{e}_n$ are the standard basis vectors of $\mathbb{R}^n$.

(a) [2 marks] If $\vec{x}$ is in $\mathbb{R}^n$, what is $T(\vec{x})$?

Solution: $T(\vec{x}) = A\vec{x} = x_1\vec{e}_n + x_2\vec{e}_{n-1} + \cdots + x_{n-1}\vec{e}_2 + x_n\vec{e}_1$. So

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_n \\ x_{n-1} \\ \dots \\ x_2 \\ x_1 \end{bmatrix}.$$

(b) [2 marks] What is A^2?

Solution:

$$A^2\vec{x} = A(A\vec{x}) = T\left(T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_2 \\ x_1 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \vec{x};$$

so $A^2 = I$, the identity matrix.

(c) [3 marks] Show that if λ is an eigenvalue of A then $\lambda = \pm 1$.

Solution: let λ be an eigenvalue of A with corresponding eigenvector $\vec{v}$. Then $\vec{v} \neq \vec{0}$ and

$$A\vec{v} = \lambda\vec{v} \Rightarrow A(A\vec{v}) = A(\lambda\vec{v}) \Rightarrow A^2\vec{v} = \lambda(A\vec{v}) \Rightarrow I\vec{v} = \lambda(\lambda\vec{v}) \Rightarrow \vec{v} = \lambda^2\vec{v},$$

so $\lambda^2 = 1$ and the result follows.

(d) [3 marks] Find the characteristic polynomial of A for $n \geq 2$.

Solution: look for the pattern. If A is 2×2, then

$$\det(xI - A) = \det\begin{pmatrix} x & -1 \\ -1 & x \end{pmatrix} = x^2 - 1;$$

if A is 3×3, then

$$\det(xI - A) = \det\begin{pmatrix} x & 0 & -1 \\ 0 & x-1 & 0 \\ -1 & 0 & x \end{pmatrix} = (x-1)(x^2-1).$$

In general, if $n = 2k$ is even then

$$\det(xI - A) = (x^2 - 1)^k;$$

if $n = 2k+1$ is odd, then

$$\det(xI - A) = (x-1)(x^2-1)^k.$$

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