University of Toronto
Faculty of Applied Science and Engineering
Final Examination, April 2020
Duration: 3 hrs
Solutions to **MAT188H1S - Linear Algebra**
Examiner: D. Burbulla

Structure of the Exam: this exam consists of four parts, Parts A through D, spread over six Assignments on Quercus:

1. Part A: Multiple Choice consists of 15 multiple choice questions, worth 2 marks each. Once you start Part A: Multiple Choice you have 50 minutes to finish it. You can only attempt each question once; and once you've done a question you cannot go back to it. Total marks for Part A: Multiple Choice: 30.

2. Part B: True or False consists of ten True or False questions, worth 1 mark each. Once you start Part B: True or False, you have 20 minutes to complete it. Total marks for Part B: True or False: 10.

   You also have to submit your counterexamples for all the True or False questions you chose as False. You can do that through the Quercus Assignment Part B: Submission.

3. Part B: Submission. This is the Assignment on Quercus to which you can submit your counterexamples for all the True or False questions that you chose as False. You can also submit your work to Crowdmark. Total marks for Part B: Submission: 5.

4. Part C: Multiple Answer Questions consists of five questions that require you to match up to 10 statements/questions with items/answers from a tear-down menu. Each question is worth 3 marks. Once you start Part C: Multiple Answer Questions, you have 30 minutes to finish it. You can only attempt each question once; and once you've done a question you cannot go back to it. Total marks for Part C: Multiple Answer Questions: 15.

5. Part D: Written Questions will give you three problems, worth 10 marks each, for which you must write up solutions to be submitted. You can only view the questions for 15 minutes. You should copy them down. You will get 1 bonus mark for viewing each question. Total marks for Part D: Written Questions : 3.

6. Part D: Submission. Once you have written up your solution to the questions from Part D—hand written please—you can upload it to Part D: Submission as a jpg, pdf or png file. Or you can submit it to Crowdmark. Each written question is worth 10 marks and part marks for each part will be indicated in the question. Total marks for Part D: Submission: 30.

Order of the Exam and Timing: you *must* complete Parts A, B and C *before* you attempt Part D. You can do Parts A, B and C in any order you wish.

1. The quizzes in Parts A, B and C have time limits, so you can spend at most 100 minutes on them. That leaves you at least 80 minutes to do the rest of the exam, including writing up your solutions and submitting them.

2. Once you start the exam, you have 3 hours to finish the whole thing.

3. Through Quercus (and Crowdmark, for that matter) we have complete records of which time you started a quiz, when you submitted an answer or a file, when you finished a quiz, etc. Thus we will know the precise instant when you started the exam, and whether or not you completed Parts A, B and C before you started Part D. We will also be able to determine if anything you submitted was late—i.e. 3 hours after you started. Late submissions will not be graded.

Submitting Your Files: for the questions that require you to show your work, you could submit your solutions to Quercus through Part B: Submission and Part D: Submission. But we have created a Crowdmark assignment as well, called Final Submissions, through which you can also submit your solutions to the written questions for Parts B and D. We would actually **prefer** if you submitted your solutions to the written questions in Parts B and D via Crowdmark. If you submit something to both Quercus and Crowdmark what you submit to each must be identical. If not, we will only mark the solution that was submitted first. No corrections or alterations to previously submitted work is permitted.

Showing Your Work and Not Showing Your Work: approximately two thirds of the exam does not require you to show your work. In the one third of the exam that requires you to show your work, in particular in Part D, you must, must explain your work. We will mark these parts very strictly: if you do not explain what you are doing, we will not try to figure it out—you will just lose marks. What you submit must be legible and coherent, making use of correct logic and correct mathematical notation. If something doesn't make sense we will just skip it. Moreover, what you submit should be your *own* work. We will watch out for solutions that are identical, or solutions that are copied verbatim from books or websites. Any such academic offence will seriously jeopardize your exam!

Notation: typing mathematical expressions into Quercus using fancy mathematical formatting is not always reliable, so for the most part the questions on Quercus will use Matlab notation:

- v = [a, b, c] is a vector, which could be a column or a row, depending on context.
- A = [a, b, c; d, e, f; g, h, i] is a matrix whose entires in the first row are a, b, c; entries in the second row are d, e, f; entries in the third row are g, h, i.
- R stands for the real numbers, $\mathbb{R}$.
- the underbar is used for subscripts and the circumflex is used for superscripts and exponentiation. For example

  ```
  x_2, A^2, A^{-1}
  ```

  mean x_2, A^2 and A^{-1}, respectively.

General Advice: put aside a 3-hour time slot in which you can focus on the exam and work without distractions or interruptions. The structure of the exam requires you to organize your time and to pay attention to one question at a time. Don't let yourself get bogged down with one question. No question in Part A, B or C is worth very much by itself, so skipping any one of those questions is not serious. There are no penalties for guessing in Parts A or C, so if you have narrowed a question down to two choices you can just follow your mathematical intuition and guess.

Technical Issues: there is not much we can do if technical issues should arise. There have been no technical issues during all the quizzes and assignments we have run through Quercus this year, nor have there been any technical issues in all the assignments we have run through Crowdmark this year. So with a little luck there won't be any technical issues during this Exam either. However, should some issue arise, try to document it in some way.

Questions During the Exam: since students may be writing the exam during a continuous 24-hour period, there is no way we can be available to everybody to answer any question that may arise during the exam. So we won't answer any. If there is something you are not clear about, you can always look it up in the book! If that doesn't help, then you can state any assumption you are making so that you can continue with the question.

Final Thoughts: this exam count for 40% of your final grade, so it is worth your effort to cooperate with the instructions, try to do as well as possible on this exam, and get it over with. You are being given a chance to complete a math course in which almost 50% of your final grade is based on work you do at home. Don't blow it!

Part A: 20 multiple choice questions, students to do 15, 2 marks each. Avg: 14.9/30

1. If U is a subspace of $\mathbb{R}^6$ and $\dim(U) = 4$, then $\dim(U^\perp)$ is

   (a) 1
   (b) $\boxed{2}$
   (c) 3
   (d) 4
   (e) 5

2. What is the rank of the matrix $\begin{bmatrix} 1 & 2 & 0 & -1 \\ -1 & 3 & 1 & 7 \\ 1 & 12 & 2 & 11 \end{bmatrix}$?

   (a) 1
   (b) $\boxed{2}$
   (c) 3
   (d) 4
   (e) 5

3. What are the eigenvalues of the matrix $A = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$?

   (a) $\boxed{\pm 1}$
   (b) ± 5
   (c) ± 3
   (d) ± 4
   (e) ± 2

4. For which value(s) of k is the set $\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ k \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

   (a) $k = 48$
   (b) $\boxed{k \neq 48}$
   (c) $k = 24$
   (d) $k \neq 24$
   (e) every $k \in \mathbb{R}$

5. Suppose $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, with $b \neq 0$. If one of the eigenvectors of A is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then another eigenvector of A is

   (a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $\boxed{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}$ (d) $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

6. Which one of the following sets is an orthogonal basis of $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ (c) $\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix} \right\}}$

(d) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \right\}$

7. Find $\vec{w}$, the projection of the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ onto the plane with equation $2x + y + 2z = 0$.

(a) $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ (b) $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ (c) $\boxed{\vec{w} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$ (d) $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (e) $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

8. For which value(s) of k is null $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 2 & k & -2 \\ 0 & -3 & 6 & k \end{bmatrix} \neq \{\vec{0}\}$?

(a) $k = \pm 4$
(b) $k = \pm 3$
(c) $\boxed{k = -4 \text{ or } k = 3}$
(d) $k = -3$ or $k = 4$
(e) $k = 3$ only

9. Which one of the following is not an orthogonal matrix?

(a) $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ (b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\boxed{\frac{1}{3} \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & 2 \\ 2 & 1 & 2 \end{bmatrix}}$ (e) $\frac{1}{13} \begin{bmatrix} 13 & 0 & 0 \\ 0 & -5 & 12 \\ 0 & 12 & 5 \end{bmatrix}$

10. $\dim \left(\text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -17 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 2 \\ 0 \end{bmatrix} \right\} \right) =$

(a) 1
(b) 2
(c) $\boxed{3}$
(d) 4
(e) 5

11. If $A = \begin{bmatrix} 1 & 0 & a & 2 \\ -2 & -1 & b & 3 \\ 3 & 1 & c & 1 \end{bmatrix}$ and the reduced row-echelon form of A is $R = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then $(a, b, c) =$

(a) $(3, -2, 0)$

(b) $\boxed{(3, -4, 7)}$

(c) $(0, 0, 1)$

(d) $(3, -2, 7)$

(e) $(1, -3, 4)$

12. Let $U = \operatorname{col} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 0 & 3 & 3 \\ 1 & -2 & 0 \end{bmatrix}$. Which one of the following sets is a basis for $U^{\perp}$?

(a) $\left\{ \begin{bmatrix} 3 \\ 3 \\ -1 \\ 0 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ (c) $\boxed{\left\{ \begin{bmatrix} 3 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}}$ (d) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

13. Suppose A is a 6×6 diagonalizable matrix with characteristic polynomial

$$C_A(x) = x^2(x-3)^2(x-4)(x-5).$$

What is the rank of A?

(a) 2

(b) 3

(c) $\boxed{4}$

(d) 5

(e) 6

14. Which one of the following statements is not true?

(a) If S is a subspace of $\mathbb{R}^5$ then $\dim(S) \leq 5$.

(b) $\boxed{\text{Every subspace of } \mathbb{R}^n \text{ has a basis.}}$

(c) If S is a set of spanning vectors for $\mathbb{R}^4$ then S contains a basis for $\mathbb{R}^4$.

(d) If S is a linearly independent set of vectors in $\mathbb{R}^7$ then S contains at most 7 vectors.

(e) If S is a spanning set of $\mathbb{R}^5$ and S contains 5 vectors, then S is a linearly independent set.

15. Let A be an $m \times n$ matrix with reduced row-echelon form R. Select all of the following statements that are True.

(a) $\boxed{\text{col}(A^T) = \text{row}(R)}$

(b) $\text{row}(A^T) = \text{col}(R)$

(c) $\text{rank}(A) + \dim(\text{null}(A^T)) = n$.

(d) $\text{row}(A^T) = \text{row}(R)$

(e) $\boxed{\text{rank}(A) + \dim(\text{null}(A^T)) = m.}$

16. Let $A = \begin{bmatrix} 4 & 13 & 7 \\ 13 & 2 & 6 \\ 7 & 6 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 1 \\ 3 & 3 & 5 \end{bmatrix}$. Which one of the following statements is true?

(a) A is diagonalizable and B is not diagonalizable.

(b) A is not diagonalizable and B is diagonalizable.

(c) $\boxed{\text{Both } A \text{ and } B \text{ are diagonalizable.}}$

(d) Neither A nor B is diagonalizable.

(e) Both A and B are diagonal.

17. Suppose A is a 3×5 matrix such that the three vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -7 \\ 5 \\ 0 \end{bmatrix}$$

span $\text{null}(A)$. Select all of the following statements that are True.

(a) $\boxed{\text{The rows of } A \text{ are linearly independent.}}$

(b) The columns of A are linearly independent.

(c) $\boxed{\text{The system } A\vec{x} = \vec{b} \text{ is consistent for every } \vec{b} \text{ in } \mathbb{R}^3.}$

(d) $\boxed{\text{The rank of } A \text{ is } 3.}$

(e) The rank of A is 2.

18. Let P be an orthogonal $n \times n$ matrix. Select all of the following statements that are true.

(a) $\boxed{P^{-1} = P^T}$

(b) $\boxed{\det(P) = \pm 1}$

(c) $\boxed{\text{The columns of } P \text{ form an orthonormal basis of } \mathbb{R}^n.}$

(d) $\boxed{\text{The rows of } P \text{ form an orthonormal basis of } \mathbb{R}^n.}$

(e) $\boxed{\|P\vec{x}\| = \|\vec{x}\|, \text{ for all vectors } \vec{x} \text{ in } \mathbb{R}^n.}$

19. Suppose A is a 3×3 matrix with eigenvalues $\lambda_1 = -3, \lambda_2 = 1, \lambda_3 = 4$ and corresponding eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Select *all* of the following statements that are true.

(a) $\boxed{\det(A) = -12}$

(b) $\boxed{A \text{ is orthogonally diagonalizable.}}$

(c) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

(d) $\boxed{A \text{ is symmetric.}}$

(e) $A^{-1} = A^T$

20. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Select all of the following statements that are true.

(a) A is symmetric.

(b) A is invertible.

(c) $\boxed{I + A \text{ is invertible, where } I \text{ is the } 3 \times 3 \text{ identity matrix.}}$

(d) $(I - A)(I + A)$ is orthogonal, where I is the 3×3 identity matrix.

(e) $\boxed{(I - A)(I + A)^{-1} \text{ is orthogonal, where } I \text{ is the } 3 \times 3 \text{ identity matrix.}}$

Part B: Fifteen True or False statements, students to do 10, one mark each. Avg: 6.4/10
Plus: students to give counterexamples for their False Choices. Avg: 1.6/5

1. If $\vec{u}$ and $\vec{v}$ are non-zero vectors in $\mathbb{R}^3$ such that $\vec{u} \cdot \vec{v} = 0$ then $\{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$ is an orthogonal basis of $\mathbb{R}^3$. **True**

2. Every invertible matrix is diagonalizable.

False: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible, sinc $\det(A) = 1 \neq 0$, but it is not diagonalizable since it only has one eigenvalue, $\lambda = 1$, repeated, but $\dim(E_1(A)) = 1 < 2$.

3. If U is a subspace of $\mathbb{R}^n$ and $\vec{v}$ is a vector in $\mathbb{R}^n$, then **True**

$$\mathrm{proj}_U\left(\mathrm{proj}_U(\vec{v})\right) = \mathrm{proj}_U(\vec{v}).$$

4. If $\vec{v}$ is a unit column vector in $\mathbb{R}^n$, then $\vec{v}^T\,\vec{v} = 1$. **True**

5. If P is a 2×2 orthogonal matrix and $\det(P) = -1$ then $I - P$ is invertible, where I is the 2×2 identity matrix.

False: let $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is orthogonal and $\det(P) = -1$, but $I - P = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is not invertible since $\det(I - P) = 0$.

6. If A is a 6×4 matrix with rank 3, then the matrix A^TA is not invertible. **True**

7. Similar matrices have the same eigenvalues and the same corresponding eigenspaces.

False: pick the reflection matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which is diagonalizable and similar to the diagonal matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so they have the same eigenvalues. But the eigenspaces of A are the lines $y = \pm x$ and the eigenspaces of D are the x and y axes.

8. If $\{\vec{u}, \vec{v}, \vec{w}\}$ is an independent set of vectors in $\mathbb{R}^4$ then so is $\{\vec{u} + \vec{v}, \vec{v} + \vec{w}, \vec{v} - \vec{w}\}$. **True**

9. If A is a diagonalizable matrix then so is A^T. **True**

10. If $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a projection onto the line with equation $y = 3x$ and $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a reflection about the line with equation $y = -2x$, then the composition $S \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a rotation of $72°$ around the origin.

False: if A is the matrix of T and B is the matrix of S, then $\det(A) = 0$ and $\det(B) = -1$. Then AB is the matrix of $S \circ T$, but $\det(AB) = \det(A)\det(B) = 0$. So $S \circ T$ can't be a rotation at all, let alone one of $72°$.

11. If $\vec{v} = \begin{bmatrix} 1 \\ m \end{bmatrix}$, then $A = \vec{v}\,(\vec{v}^T\,\vec{v})^{-1}\,\vec{v}^T$ is the matrix of a projection map. **True**

12. If the eigenvalues of A are all real numbers, then A is symmetric.

False: let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. The eigenvalues of A are real, 1 and 2, but A is not symmetric.

13. If the rows of the 3×5 matrix A are linearly independent, then $\mathrm{null}(A) = \{\vec{0}\}$.

    False: the rank of A is 3, so $\dim(\mathrm{null}(A)) = 5 - 3 = 2 \neq 0$.

14. The system of linear equations $A\vec{x} = \vec{b}$ is consistent if and only if $\mathrm{rank}(A) = \mathrm{rank}(A \,|\, \vec{b}\,)$, where $(A \mid \vec{b}\,)$ is the augmented matrix of the system $A\vec{x} = \vec{b}$. **True**

15. if $\vec{v}$ is a unit column vector in $\mathbb{R}^n$, then the matrix $\vec{v}\,\vec{v}^T$ is symmetric. **True**

Part C: Seven Multiple Answer Questions; students do 5, three marks each. Avg: 7.4/15

1. Suppose A is a 4×5 matrix that has reduced row-echelon form

$$B = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Let R_1, R_2, R_3, R_4 be the rows of A, counting from the top; let C_1, C_2, C_3, C_4, C_5 be the columns of A, counting from the left. Decide if each of the following statements Must Be True or Need Not Be True.

   (a) The rank of A is 3. Must Be True

   (b) The dimension of the null space of A is 2. Must Be True

   (c) The non-zero rows of B are a basis for the row space of A. Must Be True

   (d) $\{R_1, R_2, R_3\}$ form a basis for the row space of A. Need Not Be True

   (e) $\{C_1, C_2, C_5\}$ form a basis for the column space of A. Must Be True.

   (f) The three columns of B with the leading 1's form a basis for the column space of A. Need Not Be True

   (g) The three columns of B with the leading 1's form a basis for the column space of B. Must Be True

   (h) The null space of A is the same as the null space of B. Must Be True

   (i) The image of A is the same as the image of B. Need Not Be True.

   (j) $\{C_1, C_3\}$ do not span the column space of A. Must Be True.

2. For this question, let A be a 5×8 matrix with rank 5. Then

   (a) $\dim(\text{col}(A))$ is 5

   (b) $\dim(\text{row}(A))$ is 5

   (c) $\dim(\text{null}(A))$ is 3

   (d) $\dim\left(\text{col}(A^T)\right)$ is 5

   (e) $\dim\left(\text{row}(A^T)\right)$ is 5

   (f) $\dim\left(\text{null}(A^T)\right)$ is 0

   (g) $\dim\left((\text{null}(A))^\perp\right)$ is 5

   (h) $\dim\left(\left(\text{null}(A^T)\right)^\perp\right)$ is 5

   (i) $\dim\left(\text{null}(AA^T)\right)$ is 0

3. Decide if each of the following subsets U of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

(a) $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 \mid \ xyz = 0 \right\}$. No

(b) $U = \{\vec{x} \text{ in } \mathbb{R}^3 \mid \vec{x} \times \vec{v} = \vec{0}\}$, if $\vec{v}$ is a non-zero vector in $\mathbb{R}^3$. Yes

(c) $U = \{\vec{x} \mid T(\vec{x}) = \vec{x}\}$ if $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation. Yes

(d) $U = \{\vec{x} \mid \det[\vec{u}\ \vec{v}\ \vec{x}] = 0\}$, if $\vec{u}, \vec{v}$ are two non-parallel vectors in $\mathbb{R}^3$. Yes

(e) $U = \left\{ \begin{bmatrix} a+c \\ 2a+b-2 \\ a-b+c \end{bmatrix} \middle| \, a, b, c \text{ in } \mathbb{R} \right\}$. No

4. Decide if each of the following statements is equivalent to the statement "The $n \times n$ matrix is invertible." If it is pick Yes; it isn't, pick No.

- $\det(A) \neq 0$ Yes
- $\text{null}(A) \neq \{\vec{0}\}$ No
- $\text{im}(A) = \mathbb{R}^n$ Yes
- The columns of A form an orthogonal basis for $\mathbb{R}^n$. No
- $\lambda = 0$ is not an eigenvalue of A. Yes
- $\text{col}(A) = \text{im}(A)$ No
- The rows of A span $\mathbb{R}^n$. Yes
- A^2 is invertible. Yes
- The equation $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^n$. Yes
- For some positive number p, $A^p = I$, the $n \times n$ identity matrix. No

5. Consider the three linear transformations, $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, defined by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix}$$

$$U\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - (x_1 + x_2 + x_3)/3 \\ x_2 - (x_1 + x_2 + x_3)/3 \\ x_3 - (x_1 + x_2 + x_3)/3 \end{bmatrix}.$$

.

(a) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is the matrix of the linear transformation S

(b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is the matrix of the linear transformation T

(c) $C = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is the matrix of the linear transformation U

(d) Which of S, T or U has an inverse? S and T

(e) Which of A, B or C is diagonalizable? A and C

(f) Which of A, B, or C is orthogonally diagonalizable? A and C

(g) Which of A, B or C has eigenvalue $\lambda = 1$? A, B and C

(h) $\dim(E_1(A)) =$ 2

(i) $\dim(E_1(B)) =$ 1

(j) $\dim(E_1(C)) =$ 2

6. Let $\vec{c} \neq \vec{0}$ be a column vector in $\mathbb{R}^m$ and let $\vec{r} \neq \vec{0}$ be a row vector in $\mathbb{R}^n$; let $A = \vec{c}\,\vec{r},\ C = \vec{c}\,\vec{c}^T,\ R = \vec{r}^T\,\vec{r}$.

(a) $\mathrm{col}(A) =$ $\mathrm{span}\{\vec{c}\}$

(b) $\mathrm{row}(A) =$ $\mathrm{span}\{\vec{r}\}$

(c) $\mathrm{null}(A) =$ $\mathrm{null}(\vec{r})$

(d) $\mathrm{null}(A^T) =$ $\mathrm{null}(\vec{c}^T)$

(e) $A^T A =$ $\|\vec{c}\|^2\, R$

(f) $AA^T =$ $\|\vec{r}\|^2\, C$

(g) An eigenvector of $A^T A$ is $\vec{r}^T$

(h) An eigenvalue of $A^T A$ is $\|\vec{c}\|^2 \|\vec{r}\|^2$

(i) An eigenvector of AA^T is $\vec{c}$

(j) An eigenvalue of AA^T is $\|\vec{c}\|^2 \|\vec{r}\|^2$

7. Let $A = \begin{bmatrix} 0 & \cos\theta & 0 \\ \cos\theta & 0 & \sin\theta \\ 0 & \sin\theta & 0 \end{bmatrix}$.

(a) $\vec{u} = \begin{bmatrix} \cos\theta \\ -1 \\ \sin\theta \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue -1

(b) $\vec{v} = \begin{bmatrix} \cos\theta \\ 1 \\ \sin\theta \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue 1

(c) $\vec{w} = \begin{bmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue 0

(d) $\mathrm{tr}(A) =$ 0

(e) $\mathrm{tr}(A^2) =$ 2

(f) The eigenvalues of A^2 are 0 and 1

(g) Which value(s) of k, if any, satisfy $A^2 = kA$? none

(h) Which value(s) of k, if any, satisfy $A^3 = kA$? 1

(i) Is $P = [\vec{u}\ \vec{v}\ \vec{w}]$ an orthogonal matrix? No

(j) Is A orthogonally diagonalizable? Yes

Part D: six long written questions; students to do three of them, 10 marks each. Avg: 10.2/30

1.(a) [5 marks] Prove that if $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3\,]$ is an orthogonal 3×3 matrix then $\|\vec{a}_1 + \vec{a}_2 + \vec{a}_3\|^2 = 3$.

Proof: the columns of A form an orthonormal set, so

$$\begin{aligned}
\|\vec{a}_1 + \vec{a}_2 + \vec{a}_3\|^2 &= (\vec{a}_1 + \vec{a}_2 + \vec{a}_3) \cdot (\vec{a}_1 + \vec{a}_2 + \vec{a}_3) \\
&= \vec{a}_1 \cdot \vec{a}_1 + \vec{a}_2 \cdot \vec{a}_2 + \vec{a}_3 \cdot \vec{a}_3 + 2\vec{a}_1 \cdot \vec{a}_2 + 2\vec{a}_1 \cdot \vec{a}_3 + 2\vec{a}_2 \cdot \vec{a}_3 \\
&= \|\vec{a}_1\|^2 + \|\vec{a}_2\|^2 + \|\vec{a}_3\|^2 + 0 + 0 + 0 \\
&= 1 + 1 + 1 = 3
\end{aligned}$$

1.(b) [5 marks] Find the least squares approximating line $y = a + b\,x$ for the five data points

$$(x, y) = (-1, -2), (0, 1), (1, 2), (3, 4), (4, 4).$$

Solution: let

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix},\ \vec{z} = \begin{bmatrix} a \\ b \end{bmatrix},\ \vec{Y} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

and use the normal equations $M^T M \vec{z} = M^T Y$ to solve for a and b :

$$\begin{bmatrix} 5 & 7 \\ 7 & 27 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 24 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{86} \begin{bmatrix} 27 & -7 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 9 \\ 32 \end{bmatrix} = \frac{1}{86} \begin{bmatrix} 19 \\ 97 \end{bmatrix} \approx \begin{bmatrix} 0.22 \\ 1.13 \end{bmatrix}.$$

2.(a) [5 marks] Suppose S and T are both linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ that represent reflections in different lines passing through the origin. Prove that if $S \circ T = T \circ S$ then the two lines must be orthogonal to each other.

Proof: let S be a reflection in the line $y = mx$. The matrix of S is

$$A = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Let T be a reflection in the line $y = nx$, with $m \neq n$. The matrix of T is

$$B = \frac{1}{1+n^2}\begin{bmatrix} 1-n^2 & 2n \\ 2n & n^2-1 \end{bmatrix}.$$

Then $S \circ T = T \circ S$ if and only if $AB = BA$. Focusing in on the top right entry, say:

$$\begin{aligned} AB = BA \quad &\Rightarrow \quad 2n(1-m^2) + 2m(n^2-1) = 2m(1-n^2) + 2n(m^2-1) \\ &\Rightarrow \quad 2n - 2nm^2 + 2mn^2 - 2m = 2m - 2mn^2 + 2nm^2 - 2n \\ &\Rightarrow \quad 4n - 4m + 4mn^2 - 4nm^2 = 0 \\ &\Rightarrow \quad (n-m) + mn(n-m) = 0 \\ &\Rightarrow \quad 1 + mn = 0, \text{ since } m \neq n \\ &\Rightarrow \quad mn = -1, \end{aligned}$$

which means the lines with equations $y = mx$ and $y = nx$ are orthogonal.

2.(b) [5 marks] Find the least squares approximating line $y = a + b\,x$ for the five data points

$$(x, y) = (-2, 3), (0, 1), (1, -1), (3, 0), (5, -1).$$

Solution: let

$$M = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}, \ \vec{z} = \begin{bmatrix} a \\ b \end{bmatrix}, \ \vec{Y} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

and use the normal equations $M^T M \vec{z} = M^T Y$ to solve for a and b :

$$\begin{bmatrix} 5 & 7 \\ 7 & 39 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{146}\begin{bmatrix} 39 & -7 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -12 \end{bmatrix} = \frac{1}{146}\begin{bmatrix} 162 \\ -74 \end{bmatrix} \approx \begin{bmatrix} 1.1 \\ -0.5 \end{bmatrix}.$$

3. [10 marks] Suppose A is an orthogonally diagonalizable 3×3 matrix such that

$$E_2(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} \right\}$$

If the only other eigenvalue of A is $\lambda = -1$, find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix. Describe how you could find A, but don't calculate it.

Solution: some observations:

- We have $\dim(E_2(A)) = 2$ and $\dim(E_{-1}(A)) = 1$. This means, geometrically, that $E_2(A)$ is a plane passing through the origin and $E_{-1}(A)$ is a line passing through the origin.
- Since A is *orthogonally* diagonalizable, there is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A. In particular $E_2(A)$ and $E_{-1}(A)$ must be orthogonal complements.
- Thus $E_{-1}(A)$ is the line normal to the plane $E_2(A)$.

Let

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix};$$

let $\vec{w}$ be an eigenvector of A corresponding to the eigenvalue $\lambda = -1$. By observations above $\vec{w}$ must be a normal vector to the plane $E_2(A)$, so take

$$\vec{w} = \vec{u} \times v = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ -9 \\ -6 \end{bmatrix} \text{ or } \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix},$$

for simplicity, and we'll normalize it later. The given basis of $E_2(A)$ is not orthogonal, but we can take $\{\vec{u}, \vec{u} \times \vec{w}\}$ instead, where

$$\vec{u} \times \vec{w} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \times \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \\ -18 \end{bmatrix} \text{ or } \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix}, \text{ for simplicity.}$$

Thus an orthogonal basis of $\mathbb{R}^3$ consisting of eigenvectors of A is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix} \right\}$$

and an orthogonal matrix P that diagonalizes A is

$$P = \begin{bmatrix} 1/\sqrt{14} & 6/\sqrt{133} & 5/\sqrt{38} \\ 3/\sqrt{14} & 4/\sqrt{133} & -3/\sqrt{38} \\ -2/\sqrt{14} & 9/\sqrt{133} & -2/\sqrt{38} \end{bmatrix}, \text{ with } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Finally, if you wanted to do the calculations,

$$D = P^T AP \Rightarrow A = PDP^T.$$

4. [10 marks] Suppose A is an orthogonally diagonalizable 3×3 matrix such that

$$E_3(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$$

If the only other eigenvalue of A is $\lambda = -1$, find an orthogonal matrix P such that P^TAP is a diagonal matrix. Describe how you could find A, but don't calculate it.

Solution: some observations:

- Since A is diagonalizable we must have $\dim(E_{-1}(A)) = 2$. This means, geometrically, that $E_3(A)$ is a line passing through the origin and $E_{-1}(A)$ is a plane passing through the origin.
- Since A is *orthogonally* diagonalizable, there is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A. In particular $E_3(A)$ and $E_{-1}(A)$ must be orthogonal complements.
- Thus $E_{-1}(A) = (E_3(A))^\perp$; that is, $E_{-1}(A)$ is the plane with equation $x + y - 2z = 0$.

We have $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is a basis of $E_3(A)$. To find an orthogonal basis of $E_{-1}(A)$ take, by inspection,

$$\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}.$$

Or you can take

$$\vec{v} = \vec{u} \times \vec{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}.$$

Then $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthogonal basis of $\mathbb{R}^3$ consisting of eigenvectors of A. And an orthogonal matrix P that diagonalizes A is

$$P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & -2/\sqrt{6} \end{bmatrix}, \text{ with } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Finally, if you wanted to do the calculations,

$$D = P^TAP \Rightarrow A = PDP^T.$$

5. Let

$$U = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ -3 \\ 3 \end{bmatrix} \right\};$$

let $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Find vectors $\vec{u}$ and $\vec{v}$ such that

$$\vec{u} \in U, \vec{v} \in U^{\perp} \text{ and } \vec{x} = \vec{u} + \vec{v}.$$

Solution: let $\vec{u}_1, \vec{u}_2, \vec{u}_3$, respectively, be the given spanning vectors of U. Apply the Gram-Schmidt algorithm to find an orthogonal basis $\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$ for $U : \vec{y}_1 = \vec{u}_1$;

$$\vec{y}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{y}_1}{\|\vec{y}_1\|^2}\vec{y}_1 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}; \ \vec{y}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{y}_1}{\|\vec{y}_1\|^2}\vec{y}_1 - \frac{\vec{u}_3 \cdot \vec{y}_2}{\|\vec{y}_2\|^2}\vec{y}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$\vec{u} = \text{proj}_U(\vec{x}) = \frac{\vec{x} \cdot \vec{y}_1}{\|\vec{y}_1\|^2}\vec{y}_1 + \frac{\vec{x} \cdot \vec{y}_2}{\|\vec{y}_2\|^2}\vec{y}_2 + \frac{\vec{x} \cdot \vec{y}_3}{\|\vec{y}_3\|^2}\vec{y}_3 = \frac{4}{4}\vec{y}_1 - \frac{3}{3}\vec{y}_2 - \frac{3}{3}\vec{y}_3 = \vec{y}_1 - \vec{y}_2 - \vec{y}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Finally,

$$\vec{v} = \vec{x} - \vec{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

6. Let

$$U = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\};$$

let $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Find vectors $\vec{u}$ and $\vec{v}$ such that

$$\vec{u} \in U, \vec{v} \in U^{\perp} \text{ and } \vec{x} = \vec{u} + \vec{v}.$$

Solution: let $\vec{u}_1, \vec{u}_2, \vec{u}_3$, respectively, be the given spanning vectors of U. Apply the Gram-Schmidt algorithm to find an orthogonal basis $\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$ for $U : \vec{y}_1 = \vec{u}_1$;

$$\vec{y}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{y}_1}{\|\vec{y}_1\|^2}\vec{y}_1 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}; \ \vec{y}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{y}_1}{\|\vec{y}_1\|^2}\vec{y}_1 - \frac{\vec{u}_3 \cdot \vec{y}_2}{\|\vec{y}_2\|^2}\vec{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$\vec{u} = \text{proj}_U(\vec{x}) = \frac{\vec{x} \cdot \vec{y}_1}{\|\vec{y}_1\|^2}\vec{y}_1 + \frac{\vec{x} \cdot \vec{y}_2}{\|\vec{y}_2\|^2}\vec{y}_2 + \frac{\vec{x} \cdot \vec{y}_3}{\|\vec{y}_3\|^2}\vec{y}_3 = \frac{4}{4}\vec{y}_1 - \frac{3}{3}\vec{y}_2 - \frac{3}{3}\vec{y}_3 = \vec{y}_1 - \vec{y}_2 - \vec{y}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

Finally,

$$\vec{v} = \vec{x} - \vec{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$