University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to the **FINAL EXAMINATION, DECEMBER, 2007**
First Year - CHE, CIV, CPE, ELE, IND, LME, MEC, MMS

MAT 188H1F - LINEAR ALGEBRA
Exam Type: A

Examiners:
D. Burbulla
S. Cohen
N. Ivanov
D. Markiewicz
G. Nguyen-Chu
M. Pugh
A. Tornquist
S. Uppal

SURNAME: ____________________

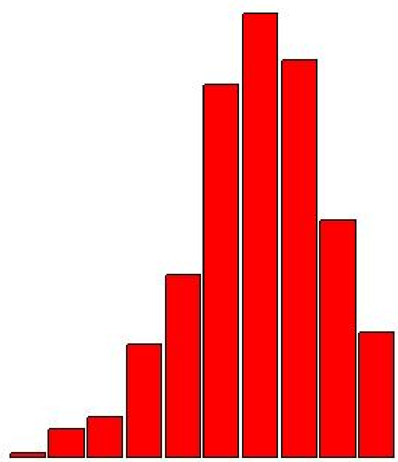
GIVEN NAMES: ____________________

STUDENT NUMBER: ____________________

SIGNATURE: ____________________

Breakdown of Results: 968 students wrote this exam. The marks ranged from 5% to 100%, and the average was 63.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	6.4%
A	18.6%	80-89%	12.2%
B	20.5%	70-79%	20.5%
C	22.8%	60-69%	22.8%
D	19.2%	50-59%	19.2%
F	18.9%	40-49%	9.4%
		30-39%	5.8%
		20-29%	2.1%
		10-19%	1.4%
		0-9%	0.2 %



1. If U is a subspace of $\mathbb{R}^6$ and dim $U = 2$, then dim $U^\perp$ is

   (a) 2

   (b) 3

   (c) 4

   (d) 5

Solution:

$$\dim U^\perp = 6 - \dim U = 6 - 2 = 4.$$

The answer is (c).

2. $\dim\left(\text{span}\left\{X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, X_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, X_5 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ -1 \end{bmatrix}\right\}\right)$ is

   (a) 2

   (b) 3

   (c) 4

   (d) 5

Solution: observe that

$$X_2 = 2X_1; X_4 = X_1 + X_3$$

and that

$$X_5 \notin \text{span}\{X_1, X_3\}.$$

Thus $\{X_1, X_3, X_5\}$ is a basis of the given subspace. The answer is (b).

3. The minimum distance from the point $P(2, 0, 3)$ to the plane with equation $x = 6$ is

   (a) 2

   (b) 4

   (c) 6

   (d) 8

Solution: a point on the plane is $P_0(6, 0, 0)$. The normal of the plane is $\vec{n} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. The minimum distance is

$$\begin{aligned} D &= \left\| \frac{\overrightarrow{P_0P} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right\| \\ &= \frac{\left| \overrightarrow{P_0P} \cdot \vec{n} \right|}{\|\vec{n}\|} \\ &= \frac{|4 + 0 + 0|}{1} \\ &= 4 \end{aligned}$$

The answer is (b).

4. If the matrix $\begin{bmatrix} 1 & a & 0 \\ 0 & 2 & 2 \\ a & 12 & 3 \end{bmatrix}$ is invertible, then

(a) $a = 3$ or $a = -3$.

(b) $a = 3$.

(c) $a \neq 3$.

(d) $a \neq 3$ and $a \neq -3$.

Solution:

$$\det \begin{bmatrix} 1 & a & 0 \\ 0 & 2 & 2 \\ a & 12 & 3 \end{bmatrix} = 2a^2 - 18.$$

The matrix is invetible $\Leftrightarrow a^2 \neq 9 \Leftrightarrow a \neq \pm 3$.
The answer is (d).

5. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Which one of the following statements is true?

(a) A is diagonalizable and B is not diagonalizable.

(b) A is not diagonalizable and B is diagonalizable.

(c) Both A and B are diagonalizable.

(d) Both A and B are not diagonalizable

Solution: B is diagonalizable, since it is symmetric. A is not diagonalizable since the eigenvalue 2 is repeated but $\dim E_2(A) = 1$. The answer is (b).

6. The equation of the plane passing through the point $(x, y, z) = (1, 0, -1)$ and containing the line $[x\ y\ z]^T = [2\ 3\ 4]^T + t[2\ 1\ 3]^T$ is

(a) $4x + 7y - 5z = 9$

(b) $2x + 3y + 4z = -2$

(c) $x - 2y + z = 0$

(d) $x - z = 6$

Solution: The normal of the plane must be

$$\vec{n} = \begin{bmatrix} 2-1 \\ 3-0 \\ 4+1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}$$

The answer is (a).

7. Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$ True or False

Solution: True

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = -4 \neq 0.$$

(b) $\operatorname{null} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \{0\}$ True or False

Solution: True.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ invertible } \Rightarrow \operatorname{null} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \{0\}.$$

(c) $\operatorname{span} \left\{ \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} \right\} = \mathbb{R}^3$ True or False

Solution: False.
$\dim \mathbb{R}^3 = 3$. So no two vectors could span $\mathbb{R}^3$.

(d) If A is a 3×3 matrix such that $\operatorname{adj} A = I$, then $A = I$. True or False

Solution: False.
Consider $A = -I$, which has $\operatorname{adj} A = I$.

(e) $\dim(\operatorname{im} A) = \dim\left(\operatorname{im} A^T\right)$ True or False

Solution: True. Both are equal to the rank of A. Or:

$$\dim(\operatorname{im} A) = \dim(\operatorname{col} A) = \dim\left(\operatorname{row} A^T\right) = \dim\left(\operatorname{col} A^T\right) = \dim\left(\operatorname{im} A^T\right)$$

8. Given that

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 2 & 0 & 3 & 0 & 1 \\ 4 & 0 & 7 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix} \text{ has reduced row-echelon form } R = \begin{bmatrix} 1 & 0 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -11 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and then find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

Solution: the rank of A is 3, the number of leading 1's in R.

subspace	description of basis	vectors in basis
row$A$	three non-zero rows of R	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 17 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$
	or three independent rows of A	$\{R_1, R_2, R_4\}$
		NB: $R_3 = 2R_1 + R_2$
col A	three independent columns of A	$\{C_1, C_3, C_4\}$
		NB: $C_5 = 17C_1 - 11C_3 + 4C_4$
null A	two basic solutions to $AX = 0$	$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -17 \\ 0 \\ 11 \\ -4 \\ 1 \end{bmatrix} \right\}$

9. Show that

$$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, \det \begin{bmatrix} x_1 & 0 & 3 & -1 \\ x_2 & 1 & 0 & 1 \\ x_3 & 0 & 0 & 1 \\ x_4 & -1 & 1 & 1 \end{bmatrix} = 0 \right\}$$

is a subspace and find a basis of U.

***U* is a subspace: Method 1.** Let C_{ij} be the (i,j)-cofactor of A. Then

$$\begin{aligned} U &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, x_1 C_{11} + x_2 C_{21} + x_3 C_{31} + x_4 C_{41} = 0 \right\} \\ &= \operatorname{null} \begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} \end{bmatrix} \\ &= \operatorname{null} \begin{bmatrix} -1 & 3 & -7 & 3 \end{bmatrix}. \end{aligned}$$

So U is a subspace, and $\dim U = 4 - 1 = 3$.

***U* is a subspace: Method 2.**

$$\det \begin{bmatrix} x_1 & 0 & 3 & -1 \\ x_2 & 1 & 0 & 1 \\ x_3 & 0 & 0 & 1 \\ x_4 & -1 & 1 & 1 \end{bmatrix} = 0$$

$$\Leftrightarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is linearly dependent}$$

$$\Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \text{ since these three are independent}$$

$$\Leftrightarrow U = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Basis of *U*, from Method 1 or Method 2:

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

10. Let

$$A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

be the matrix of the projection onto the line $y = 2x$. Find the eigenvalues and eigenvectors of A, and interpret your results geometrically.

Solution:

$$\begin{aligned} \det\begin{bmatrix} \lambda - 1/5 & -2/5 \\ -2/5 & \lambda - 4/5 \end{bmatrix} &= \lambda^2 - \lambda + \frac{4}{5} - \frac{4}{5} \\ &= \lambda^2 - \lambda \\ &= \lambda(\lambda - 1) \end{aligned}$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 0$.

Eigenspaces:

$$E_1(A) = \text{null}\begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}.$$

$$E_0(A) = \text{null}\begin{bmatrix} -1/5 & -2/5 \\ -2/5 & -4/5 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}.$$

Geometric Interpretation: let $\vec{d_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \vec{d_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$E_1(A)$ is the line $y = 2x$. For every vector $\vec{v} \in E_1(A)$, $\vec{v}$ is parallel to the line $y = 2x$ and so

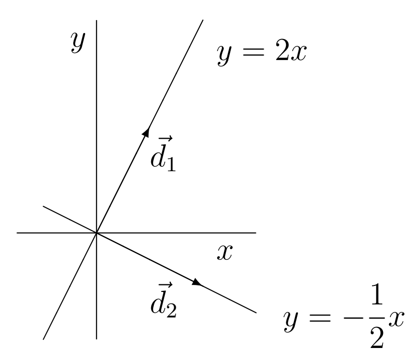
$$A\vec{v} = \text{proj}_{\vec{d_1}}\vec{v} = \vec{v}.$$

$E_0(A)$ is the line

$$y = -\frac{1}{2}x,$$

which is perpendicular to $E_1(A)$. For every vector $\vec{v} \in E_0(A)$, $\vec{v}$ is perpendicular to the line $y = 2x$ and so

$$A\vec{v} = \text{proj}_{\vec{d_1}}\vec{v} = \vec{0}.$$



11. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Solution:

Step 1: Find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & 1 & -1 \\ 1 & \lambda - 1 & 1 \\ -1 & 1 & \lambda - 1 \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda - 1 & 1 & -1 \\ \lambda & \lambda & 0 \\ -\lambda & 0 & \lambda \end{bmatrix} \\ &= \lambda^2 \det \begin{bmatrix} \lambda - 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ &= \lambda^2(\lambda - 1 - 1 - 1) \\ &= \lambda^2(\lambda - 3) \end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 0$.

Step 2: Find mutually orthogonal eigenvectors of A.

$$E_3(A) = \text{null} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$$E_0(A) = \text{null} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put in the columns of P. So

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

12. Let $U = \operatorname{span}\left\{X_1 = [\begin{array}{cccc} 0 & 1 & 0 & 1 \end{array}]^T, X_2 = [\begin{array}{cccc} 1 & 0 & 0 & 1 \end{array}]^T, X_3 = [\begin{array}{cccc} 1 & -1 & 0 & 0 \end{array}]^T\right\}$; let $X = \left[\begin{array}{cccc} 1 & 1 & 0 & 1 \end{array}\right]^T$. Find:

(a) an orthogonal basis of U.

Solution: Use the Gram-Schmidt algorithm.

$$F_1 = X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$F_2 = X_2 - \frac{X_2 \cdot F_1}{\|F_1\|^2} F_1 = X_2 - \frac{1}{2} F_1 = \begin{bmatrix} 1 \\ -1/2 \\ 0 \\ 1/2 \end{bmatrix}; \text{ take } F_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_3 \cdot F_2}{\|F_2\|^2} F_2 = X_3 + \frac{1}{2} F_1 - \frac{1}{2} F_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(Aside: this is not wrong. It means that X_3 is actually a linear combination of X_1 and X_2. In fact: $X_3 = X_2 - X_1$, and so $\dim U = 2$. You could have started simply with X_1 and X_2.) Thus an orthogonal basis of U is

$$\left\{ F_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, F_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) $\operatorname{proj}_U(X)$.

Solution:

$$\operatorname{proj}_U X = \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \frac{X \cdot F_2}{\|F_2\|^2} F_2 = \frac{2}{2} F_1 + \frac{2}{6} F_2 = F_1 + \frac{1}{3} F_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ 0 \\ 4/3 \end{bmatrix}.$$

13. Find the least squares approximating quadratic for the data points

$$(-1,0),(0,4),(1,1),(1,-2).$$

Solution: Let the least squares approximating quadratic be $y = a+bx+cx^2$; let

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, Z = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, Y = \begin{bmatrix} 0 \\ 4 \\ 1 \\ -2 \end{bmatrix}.$$

Solve the normal equations.

$$\begin{aligned}
M^TMZ = M^TY \quad &\Leftrightarrow \quad \begin{bmatrix} 4 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 & 0 & -8 \\ 0 & 3 & -1 \\ -8 & -1 & 11 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 16 \\ -1 \\ -17 \end{bmatrix}
\end{aligned}$$

So the least squares approximating quadratic is

$$y = 4 - \frac{1}{4}x - \frac{17}{4}x^2.$$

Aside: the calculations to find $(M^TM)^{-1}$ were

$$\begin{aligned}
[M^TM|I] = \left[\begin{array}{ccc|ccc} 4 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad &\to \quad \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 1 & 0 \\ 0 & -11 & -1 & 1 & -4 & 0 \\ 0 & -8 & 0 & 0 & -3 & 1 \end{array}\right] \\
&\to \quad \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3/8 & -1/8 \\ 0 & 0 & 1 & -1 & -1/8 & 11/8 \end{array}\right] \\
&\to \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 3/8 & -1/8 \\ 0 & 0 & 1 & -1 & -1/8 & 11/8 \end{array}\right] = [I|(M^TM)^{-1}]
\end{aligned}$$