University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2008**
First Year - CHE, CIV, CPE, ELE, IND, LME, MEC, MMS

MAT188H1F - LINEAR ALGEBRA
Exam Type: A

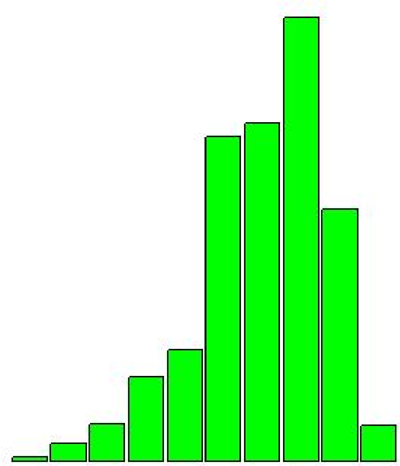
General Comments:

1. Most of this exam was very similar to last year's exam. To be specific, the types of questions used for Questions 1, 2, 3, 4, 5, 6, 7, 8, 11, 12 and 13 were identical on both exams. In particular, for Questions 8, 11, 12 and 13 only the numbers were changed. Anybody who studied last year's exam should have aced these four questions.

2. Only Questions 9 and 10 of this year's exam represented different types of questions, compared to last year's exam. Question 10 could be done in lots of different ways, and many students got it perfect. Question 9 had two parts. Part (a) was right out of the textbook; indeed you could have quoted the textbook to the effect that similar matrices have the same eigenvalues. Part (b) of Question 9 was the only real challenging part of this exam. Very few students got it!

Breakdown of Results: 839 students wrote this exam. The marks ranged from 8% to 98%, and the average was 64.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.1%
A	17.4%	80-89%	15.2%
B	26.9%	70-79%	26.9%
C	20.5%	60-69%	20.5%
D	19.7%	50-59%	19.7%
F	15.5%	40-49%	6.8%
		30-39%	5.1%
		20-29%	2.3 %
		10-19%	1.1%
		0-9%	0.2%



1. If U is a subspace of $\mathbb{R}^7$ and $\dim U = 3$, then $\dim U^\perp$ is

   (a) 3

   (b) 4

   (c) 5

   (d) 6

   Solution:

   $$\dim U^\perp = 7 - \dim U = 7 - 3 = 4.$$

   The answer is (b).

2. $\dim \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, 5x_1 + 3x_2 - x_3 + 2x_4 = 0 \right\}$ is

   (a) 1

   (b) 2

   (c) 3

   (d) 4

   Solution:

   $$\dim\left(\text{null}\begin{bmatrix} 5 & 3 & -1 & 2 \end{bmatrix}\right) = 4 - 1 = 3.$$

   The answer is (c).

3. The minimum distance from the point $P(2, 0, 3)$ to the plane with equation $x+2y+2z = 2$ is

   (a) 1

   (b) 2

   (c) 5/3

   (d) 2/3

   Solution:

   $$D = \frac{|2 + 0 + 2(3) - 2|}{\sqrt{1 + 2^2 + 2^2}} = \frac{6}{3} = 2$$

   The answer is (b).

4. The values of a for which the matrix $\begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & 4 \end{bmatrix}$ is not invertible, are

(a) $a = 2$ or $a = -2$.

(b) $a = 2$ only

(c) $a \neq 2$.

(d) $a \neq 2$ and $a \neq -2$.

Solution:

$$\det \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & 4 \end{bmatrix} = 8 - 2a^2 = 2(4 - a^2) = 0 \Leftrightarrow a = \pm 2.$$

The answer is (a).

5. Let s and t be parameters, and consider the two lines with vector equations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -6 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

Which one of the following statements is true?

(a) The two lines are parallel and do not intersect.
(b) The two lines intersect and are perpendicular to each other.
(c) The two lines do not intersect and are not parallel.
(d) The two lines intersect and are not perpendicular to each other.

Solution:

$$\begin{bmatrix} 4 \\ 1 \\ -6 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow t = -1, s = 2; \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 0.$$

So the lines intersect at $(6, 1, 0)$ and are perpendicular. The answer is (b).

6. Suppose $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 7y \\ 3x + 5y \end{bmatrix}$. Then the area of the image of the unit square is

(a) 10

(b) 21

(c) 11

(d) -11

Solution:

$$\left|\det \begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}\right| = |10 - 21| = 11.$$

The answer is (c).

7. Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) $U = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\}$ is a linearly independent set of vectors in $\mathbb{R}^3$. **True** or **False**

False: $\dim \mathbb{R}^3 = 3$, so no four vectors in $\mathbb{R}^3$ can be linearly independent.

(b) If A is a 7×5 matrix and $\dim(\text{null}\, A) = 3$, then $\dim(\text{im}\, A) = 2$. **True** or **False**

True: $5 = \dim(\text{null}\, A) + \dim(\text{im}\, A) \Rightarrow \dim(\text{im}\, A) = 5 - 3 = 2$.

(c) $\text{span} \left\{ \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3$ **True** or **False**

True: $\det \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 1 \\ -1 & 7 & 1 \end{bmatrix} = 4 - 2 + 21 + 1 - 28 - 6 = -10 \neq 0$,
so the matrix is invertible, which means its columns span $\mathbb{R}^3$.

(d) The composition of a reflection in the line $y = 6x$ and a projection onto the line $y = -3x$ is invertible. **True** or **False**

False: If A is the matrix of the reflection and B is the matrix of the projection, then $\det A = -1$ and $\det B = 0$. The matrix of the composition is AB, and $\det(AB) = \det A \det B = 0$. Thus the composition is not invertible. Note: the result is the same regardless of the order of the composition.

(e) Every invertible matrix is diagonalizable. **True** or **False**

False: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible since $\det A = 1 \neq 0$, but it is not diagonalizable since $\lambda = 1$ is a repeated eigenvalue and $\dim E_1(A) = 1 < 2$.

8. Given that

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & -1 \\ 2 & 2 & 3 & 0 & 7 \\ 1 & 1 & 7 & 2 & -1 \\ 0 & 0 & 5 & 1 & 0 \end{bmatrix} \text{ has reduced row-echelon form } R = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and then find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

Solution: the rank of A is 3, the number of leading 1's in R.

subspace	description of basis	vectors in basis
row$A$	three non-zero rows of R	$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$
	any three independent rows of A	$\{R_1, R_2, R_3\}$ $\{R_1, R_2, R_4\}$ $\{R_2, R_3, R_4\}$ NB: $R_4 = R_3 - R_1$
col A	three independent columns of A	$\{C_1, C_3, C_4\}$ NB: $C_2 = C_1; C_5 = 2C_1 + C_3 - 5C_4$
null A	two basic solutions to $AX = 0$	$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 5 \\ 1 \end{bmatrix} \right\}$

9(a) [5 marks] Suppose A and B are two $n \times n$ matrices and that there is an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Show that A and B have the same eigenvalues but not necessarily the same eigenvectors.

Solution: STATE: A and B are similar matrices, which, by a Theorem in the book, have the same eigenvalues. OR CALCULATE: let λ be an eigenvalue of B with corresponding (non-zero) eigenvector X. Then

$$\begin{aligned} BX = \lambda X &\Rightarrow (P^{-1}AP)X = \lambda X \\ &\Rightarrow APX = P(\lambda X) \\ &\Rightarrow A(PX) = \lambda(PX) \end{aligned}$$

Since P is invertible and $X \neq O$, PX is non-zero as well. Thus λ is also an eigenvalue of A, with corresponding eigenvector PX.

Eigenvectors for such matrices need not be the same. As an example, consider

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ which both have eigenvalues } \pm 1.$$

B is the diagonalization of A, so A and B are similar. B has eigenvectors $\vec{\imath}$ and $\vec{\jmath}$, but A has eigenvectors $\vec{\imath} + \vec{\jmath}$ and $\vec{\imath} - \vec{\jmath}$.

9(b) [5 marks] Suppose A and B are both $n \times n$ matrices with n distinct eigenvalues, such that $AB = BA$. Show that A and B have the same eigenvectors, but not necessarily the same eigenvalues.

Solution: This is quite tricky. Let λ be an eigenvalue of A with corresponding eigenvector X.

$$\begin{aligned} AX = \lambda X &\Rightarrow BAX = B(\lambda X) = \lambda BX \\ (\text{use } AB = BA) &\Rightarrow ABX = \lambda BX \\ &\Rightarrow BX \in E_\lambda(A) \\ &\Rightarrow BX \in \operatorname{span}\{X\}, \text{ since } \dim E_\lambda(A) = 1 \\ &\Rightarrow BX = \mu X, \text{ for some scalar } \mu \end{aligned}$$

So X is also an eigenvector of B.

Eigenvalues for such matrices need not be the same. As an example, consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \text{ for which } AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}.$$

Both have eigenvectors $\vec{\imath}$ and $\vec{\jmath}$; but they have totally different eigenvalues.

10. Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Find the matrix of T.

Solutions: There are lots of ways to do this question. Here is the shortest. Let the matrix of T be A. Then

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \text{ and } A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Combine the above into one equation and solve for A :

$$\begin{aligned}
A\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 3 & 4 \\ 6 & -2 \end{bmatrix} \\
\Leftrightarrow A &= \begin{bmatrix} 3 & 4 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \\
\Leftrightarrow A &= \begin{bmatrix} 3 & 4 \\ 6 & -2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \\
\Leftrightarrow A &= \frac{1}{5}\begin{bmatrix} -5 & 10 \\ 10 & 10 \end{bmatrix} \\
\Leftrightarrow A &= \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}
\end{aligned}$$

Comments: Note that A is symmetric. This is not a coincidence, since

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

That is, $\mathbb{R}^2$ has an orthogonal basis of eigenvectors of A, so A must be symmetric; and hence diagonalizable. This leads to **another way** to find A :

$$\begin{aligned}
\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} A \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad &\Leftrightarrow \quad A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \\
&\Leftrightarrow \quad A = \begin{bmatrix} 3 & 4 \\ 6 & -2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \\
&\Leftrightarrow \quad A = \frac{1}{5}\begin{bmatrix} -5 & 10 \\ 10 & 10 \end{bmatrix}
\end{aligned}$$

11. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution:

Step 1: Find the eigenvalues of A.

$$\begin{aligned}
\det(\lambda I - A) &= \det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} \\
&= \det \begin{bmatrix} \lambda+1 & -1-\lambda & 0 \\ -1 & \lambda & -1 \\ 0 & -1-\lambda & \lambda+1 \end{bmatrix} \\
&= (\lambda+1)^2 \det \begin{bmatrix} 1 & -1 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & 1 \end{bmatrix} \\
&= (\lambda+1)^2(\lambda - 1 - 1) \\
&= (\lambda+1)^2(\lambda - 2)
\end{aligned}$$

So the eigenvalues of A are $\lambda_1 = -1$ repeated, and $\lambda_2 = 2$.

Step 2: Find mutually orthogonal eigenvectors of A.

$$E_2(A) = \text{null} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$E_{-1}(A) = \text{null} \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

12. Let $U = \operatorname{span}\left\{X_1 = [\ 0\ \ 1\ \ 0\ \ 1\ ]^T, X_2 = [\ 1\ \ 0\ \ 0\ \ 1\ ]^T, X_3 = [\ 0\ \ 0\ \ 1\ \ 1\ ]^T\right\}$; let $X = \left[\ 1\ \ 1\ \ 0\ \ 1\ \right]^T$. Find:

(a) [6 marks] an orthogonal basis of U.

Solution: Use the Gram-Schmidt algorithm.

$$F_1 = X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}; F_2 = X_2 - \frac{X_2 \cdot F_1}{\|F_1\|^2} F_1 = X_2 - \frac{1}{2} F_1 = \frac{1}{2} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix};$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_3 \cdot F_2}{\|F_2\|^2} F_2 = X_3 - \frac{1}{2} F_1 - \frac{1}{6} F_2 = \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Optional: clear fractions and take

$$F_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, F_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, F_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Either way $\{F_1, F_2, F_3\}$ is an orthogonal basis of U.

(b) [6 marks] $\operatorname{proj}_U(X)$.

Solution:

$$\begin{aligned} \operatorname{proj}_U X &= \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \frac{X \cdot F_2}{\|F_2\|^2} F_2 + \frac{X \cdot F_3}{\|F_3\|^2} F_3 \\ &= \frac{2}{2} F_1 + \frac{2}{6} F_2 - \frac{1}{12} F_3 \\ &= \frac{1}{4} \begin{bmatrix} 3 \\ 3 \\ -1 \\ 5 \end{bmatrix}. \end{aligned}$$

Cross-check or **Alternate Solution:** $U^\perp = \operatorname{span}\{Y\}$ with $Y = [\ 1\ \ 1\ \ 1\ \ -1\ ]^T$. Then

$$\operatorname{proj}_U X = X - \operatorname{proj}_{U^\perp} X = X - \frac{X \cdot Y}{\|Y\|^2} Y = X - \frac{1}{4} Y = \frac{1}{4} \begin{bmatrix} 3 \\ 3 \\ -1 \\ 5 \end{bmatrix}.$$

13. Find the least squares approximating quadratic for the data points

$$(-1,0),(0,2),(1,1),(3,-2).$$

Solution: Let the least squares approximating quadratic be $y = a + bx + cx^2$; let

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}, Z = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, Y = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -2 \end{bmatrix}.$$

Solve the normal equation.

$$\begin{aligned}
M^T M Z = M^T Y \quad &\Leftrightarrow \quad \begin{bmatrix} 4 & 3 & 11 \\ 3 & 11 & 27 \\ 11 & 27 & 83 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -17 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 & 3 & 11 \\ 3 & 11 & 27 \\ 11 & 27 & 83 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -5 \\ -17 \end{bmatrix} \\
\text{(See below for calculations.)} \quad &\Leftrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{440} \begin{bmatrix} 184 & 48 & -40 \\ 48 & 211 & -75 \\ -40 & -75 & 35 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ -17 \end{bmatrix} \\
&\Leftrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{110} \begin{bmatrix} 156 \\ 67 \\ -65 \end{bmatrix}
\end{aligned}$$

So the least squares approximating quadratic is

$$y = \frac{78}{55} + \frac{67}{110}x - \frac{13}{22}x^2.$$

Aside: The adjoint method was used to find $(M^T M)^{-1}$. You need your calculator! $\det(M^T M) = 4 \cdot 11 \cdot 83 + 3 \cdot 27 \cdot 11 + 11 \cdot 3 \cdot 27 - 11 \cdot 11 \cdot 11 - 27 \cdot 27 \cdot 4 - 3 \cdot 3 \cdot 83 = 440$, and

$$\begin{aligned}
\operatorname{adj}(M^T M) &= \begin{bmatrix} 11 \cdot 83 - 27 \cdot 27 & -(3 \cdot 83 - 11 \cdot 27) & 3 \cdot 27 - 11 \cdot 11 \\ -(3 \cdot 83 - 27 \cdot 11) & 4 \cdot 83 - 11 \cdot 11 & -(4 \cdot 27 - 11 \cdot 3) \\ 3 \cdot 27 - 11 \cdot 11 & -(4 \cdot 27 - 11 \cdot 3) & 4 \cdot 11 - 3 \cdot 3 \end{bmatrix}^T \\
&= \begin{bmatrix} 184 & 48 & -40 \\ 48 & 211 & -75 \\ -40 & -75 & 35 \end{bmatrix}.
\end{aligned}$$

So

$$(M^T M)^{-1} = \frac{1}{440} \begin{bmatrix} 184 & 48 & -40 \\ 48 & 211 & -75 \\ -40 & -75 & 35 \end{bmatrix}.$$