University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2011**
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

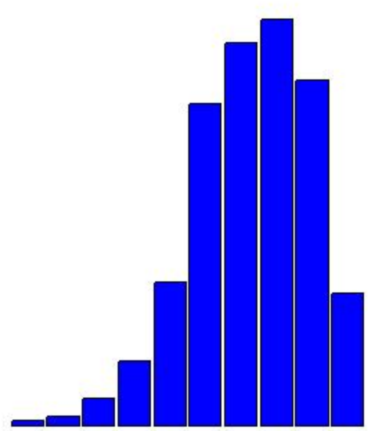
MAT188H1F - LINEAR ALGEBRA
Exam Type: A

General Comments:

1. In Question 2(a), few students realized that the zero vector is in *every* subspace of $\mathbb{R}^n$.
2. Questions 3, 6 and 7 were almost identical to analogous questions from last year's exam; these questions should have been aced.
3. In Question 4 very few students could actually draw the image of the unit square; and many used row-reduction (incorrectly!) to find the inverse of the matrix of T, instead of using the simple formula for the inverse of a 2×2 matrix.
4. In Question 5, many students calculated $C_R(x)$, but didn't use the quadratic formula to find the eigenvalues of R. The point of part (b) was to show geometrically why the only real solutions to $C_R(x) = 0$ are $x = \pm 1$.

Breakdown of Results: 947 students wrote this exam. The marks ranged from 1% to 100%, and the average was 67.0%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	7.2%
A	26.0%	80-89%	18.8%
B	22.1%	70-79%	22.1%
C	20.8%	60-69%	20.8%
D	17.5%	50-59%	17.5%
F	13.6%	40-49%	7.8%
		30-39%	3.5%
		20-29%	1.5%
		10-19%	0.5%
		0-9%	0.3%



1. Find the following:

(a) [6 marks] the values of a for which the matrix $A = \begin{bmatrix} 1 & a & 0 \\ 2 & 0 & a \\ a & -1 & 1 \end{bmatrix}$ is not invertible.

Solution: $\det \begin{bmatrix} 1 & a & 0 \\ 2 & 0 & a \\ a & -1 & 1 \end{bmatrix} = a^3 - a = 0 \Leftrightarrow a = 0$ or $a = \pm 1$.

(b) [6 marks] the minimum distance between the skew lines $\mathbb{L}_1$ and $\mathbb{L}_2$

$$\mathbb{L}_1 : \begin{cases} x &= 1 + t \\ y &= 0 - t \\ z &= 1 + 3t \end{cases} ; \quad \mathbb{L}_2 : \begin{cases} x &= 2 - s \\ y &= 3 - s \\ z &= 1 + s \end{cases}$$

where s and t are parameters.

Solution: the minimum distance is $D = \left\| \operatorname{proj}_{\vec{n}} \overrightarrow{PQ} \right\|$ with

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} \text{ and } \overrightarrow{PQ} = \begin{bmatrix} 2-1 \\ 3-0 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix};$$

so $D = |\overrightarrow{PQ} \cdot \vec{n}| / \|\vec{n}\| = |2 - 12| / \sqrt{24} = 5/\sqrt{6}$.

(c) [6 marks] a basis for $U^\perp$ if

$$U = \operatorname{span}\left\{ [\ 1\ \ 0\ \ 1\ \ 0\ \ 1\ ]^T, [\ 0\ \ 1\ \ 2\ \ -3\ \ 0\ ]^T, [\ 1\ \ 1\ \ 1\ \ 1\ \ 1\ ]^T \right\}.$$

Solution: $U^\perp =$

$$\operatorname{null} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & -3 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \operatorname{null} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{bmatrix} = \operatorname{null} \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} -2s - t \\ -s \\ 2s \\ s \\ t \end{bmatrix} \middle| \ s, t \in \mathbb{R} \right\}; \text{ so a basis for } U^\perp \text{ is } \left\{ \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(d) [6 marks] the scalar equation of the plane passing through the three points

$$P(-2, 3, 5),\ Q(2, 2, 1),\ R(2, 0, 0).$$

Solution: $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 4 & -1 & -4 \end{bmatrix}^T \times \begin{bmatrix} 4 & -3 & -5 \end{bmatrix}^T = \begin{bmatrix} -7 & 4 & -8 \end{bmatrix}^T$; so the equation is $-7x + 4y - 8z = -14 + 0 + 0 \Leftrightarrow 7x - 4y + 8z = 14$.

2. [2 marks each] Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) An $n \times n$ matrix A is not invertible if and only if $\operatorname{col}(A)$ contains the zero vector.

True or **False**

> **False:** the zero vector is in *every* subspace, so $\operatorname{col}(A)$ *always* contains the zero vector, regardless of A. In particular, the identity matrix I *is* invertible, but $\operatorname{col}(I)$ contains the zero vector.

(b) $\dim\left(\operatorname{im}\begin{bmatrix} 5 & 1 & -1 & 2 \\ 3 & 6 & 1 & 1 \\ 2 & -5 & -2 & 1 \end{bmatrix}\right) = 2$ **True** or **False**

> **True:** $\dim(\operatorname{im}(A)) = r$, where r is the rank of A. The matrix
>
> $$\begin{bmatrix} 5 & 1 & -1 & 2 \\ 3 & 6 & 1 & 1 \\ 2 & -5 & -2 & 1 \end{bmatrix}$$
>
> has rank 2 because it only has 2 independent rows: $R_1 = R_2 + R_3$.

(c) If U is a subspace of $\mathbb{R}^6$ and $\dim U = 2$ then $\dim U^\perp = 4$. **True** or **False**

> **True:** $6 = \dim U + \dim U^\perp \Rightarrow \dim U^\perp = 6 - 2 = 4$.

(d) If $\lambda \neq 0$ is an eigenvalue of A, then $\dfrac{\det(A)}{\lambda}$ is an eigenvalue of $\operatorname{adj}(A)$.

True or **False**

> **True:** $AX = \lambda X \Rightarrow \operatorname{adj}(A)(AX) = \operatorname{adj}(A)(\lambda X) \Rightarrow \det(A)X = \lambda \operatorname{adj}(A)X$

(e) If the eigenvalues of A are all real, then A is symmetric. **True** or **False**

> **False:** the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ are real, but A is not symmetric.

3. [10 marks] Given that the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 6 & 6 & 0 & 0 \\ -1 & 4 & 3 & -1 & -1 \\ 1 & -1 & 0 & 1 & 2 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

Solution: the rank of A is $r = 3$, the number of leading 1's in R.

U	$\dim U$	description of basis	vectors in basis
row$A$	$r = 3$	three non-zero rows of R	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
		three independent rows of A	$\{R_1, R_2, R_4\}$ $\{R_1, R_3, R_4\}$ $\{R_2, R_3, R_4\}$ NB: $R_3 = R_2 - R_1$
col A	$r = 3$	three independent columns of A	$\{C_1, C_2, C_5\}$ NB: $C_3 = C_1 + C_2; C_4 = C_1$
		not three independent columns of R	col$R \neq$ col$A$
null A	$5 - r = 2$	two basic solutions to $AX = 0$	$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

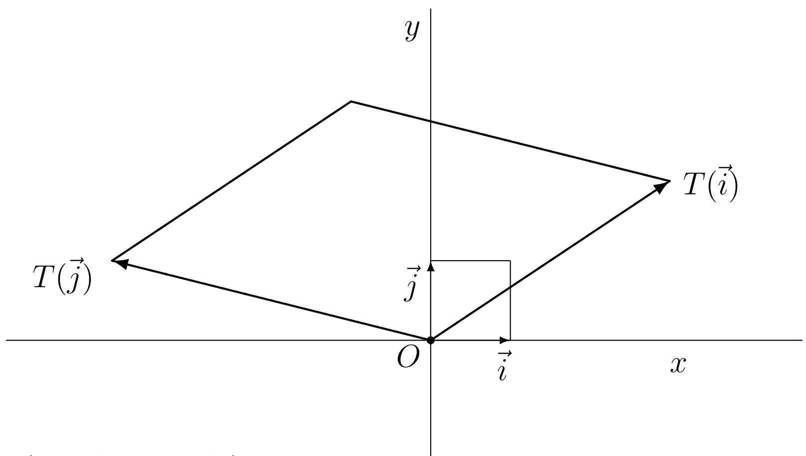
4. [10 marks] Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x - 4y \\ 2x + y \end{bmatrix}.$$

(a) [5 marks] Draw the image under T of the unit square, and calculate its area.

Solution: the image of the unit square is the parallelogram determined by

$$T(\vec{i}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } T(\vec{j}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$



Its area is $\left|\det \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}\right| = 3 + 8 = 11.$

(b) [5 marks] Find the formula for $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

Solution: the matrix of T is $A = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$ and the matrix of T^{-1} is

$$A^{-1} = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix};$$

so

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{11}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} x + 4y \\ -2x + 3y \end{bmatrix}.$$

5. [10 marks] Let $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) [5 marks] Find the eigenvalues of R.

Solution: $\det \begin{bmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{bmatrix} =$

$$(\lambda - \cos\theta)^2 + \sin^2\theta = \lambda^2 - 2\lambda\,\cos\theta + \cos^2\theta + \sin^2\theta = \lambda^2 - 2\lambda\,\cos\theta + 1.$$

Use the quadratic formula:

$$\lambda^2 - 2\lambda\,\cos\theta + 1 = 0 \Leftrightarrow \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-\sin^2\theta},$$

which you can express as $\lambda = \cos\theta \pm i\,\sin\theta$, if you like.

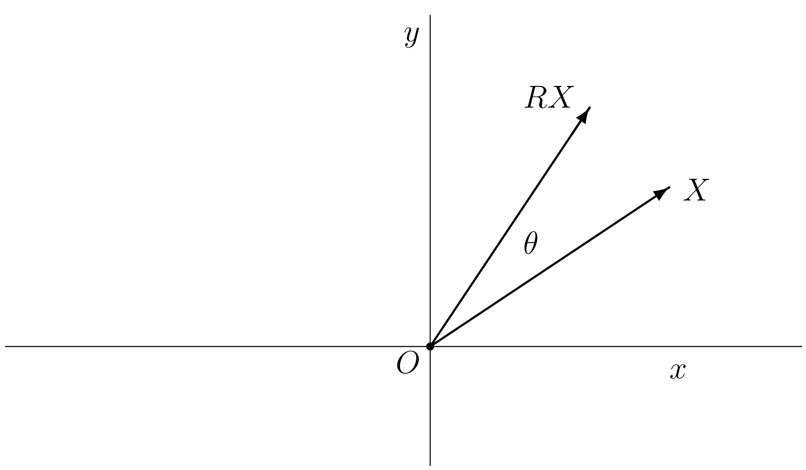
(b) [5 marks] Show geometrically that R has eigenvalues in $\mathbb{R}$ and eigenvectors in $\mathbb{R}^2$ only if θ is an integral multiple of π. What are the eigenvalues?

Solution: let X be an eigenvector of R with corresponding eigenvalue $\lambda \neq 0$. So $RX = \lambda X$, which means RX and X are parallel vectors. On the other hand, R is the rotation matrix so RX is the result of rotating X around the origin by θ, counterclockwise. Then,

1. $RX = X$ and $\lambda = 1$ means $\theta = 0$, or any even multiple of π;
2. $RX = -X$ and $\lambda = -1$ means $\theta = \pi$, or any odd multiple of π.

where we have also used the fact that $\|RX\| = \|X\|$.

Or draw a picture:



For RX to be parallel to X and in the same direction as X, $\theta = 0$ or any even multiple of π. Then from part (a), $\lambda = 1$.

For RX to be parallel to X and in the opposite direction as X, $\theta = \pi$ or any odd multiple of π. Then from part (a), $\lambda = -1$.

6. Let $U = \operatorname{span}\left\{X_1 = [\ 0\ \ 1\ \ -1\ \ 0\ ]^T, X_2 = [\ 2\ \ 0\ \ 0\ \ -1\ ]^T, X_3 = [\ 1\ \ 1\ \ 0\ \ -1\ ]^T\right\}$; let $X = \left[\ 1\ \ 1\ \ 0\ \ 1\ \right]^T$. Find:

(a) [6 marks] an orthogonal basis of U.

Solution: since $X_1 \cdot X_2 = 0$ already, you only need to use the Gram-Schmidt algorithm to find F_3. Take $F_1 = X_1,\ F_2 = X_2$, and

$$F_3 = X_3 - \frac{X_3 \cdot X_1}{\|X_1\|^2} X_1 - \frac{X_3 \cdot X_2}{\|X_2\|^2} X_2 = X_3 - \frac{1}{2} X_1 - \frac{3}{5} X_2 = \frac{1}{10} \begin{bmatrix} -2 \\ 5 \\ 5 \\ -4 \end{bmatrix}.$$

Optional: clear fractions and take

$$F_3 = \begin{bmatrix} -2 \\ 5 \\ 5 \\ -4 \end{bmatrix}.$$

Either way $\{F_1, F_2, F_3\}$ is an orthogonal basis of U.

(b) [6 marks] $\operatorname{proj}_U(X)$.

Solution: using $\{F_1, F_2, F_3\}$ with fractions cleared.

$$\begin{aligned} \operatorname{proj}_U X &= \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \frac{X \cdot F_2}{\|F_2\|^2} F_2 + \frac{X \cdot F_3}{\|F_3\|^2} F_3 \\ &= \frac{1}{2} F_1 + \frac{1}{5} F_2 - \frac{1}{70} F_3 \\ &= \frac{1}{7} \begin{bmatrix} 3 \\ 3 \\ -4 \\ -1 \end{bmatrix}. \end{aligned}$$

Cross-check/Alternate Solution: $U^\perp = \operatorname{span}\{Y\}$ with $Y = [\ 1\ \ 1\ \ 1\ \ 2\ ]^T$. Then

$$\operatorname{proj}_U X = X - \operatorname{proj}_{U^\perp}(X) = X - \frac{X \cdot Y}{\|Y\|^2} Y = X - \frac{4}{7} Y = \frac{1}{7} \begin{bmatrix} 3 \\ 3 \\ -4 \\ -1 \end{bmatrix}.$$

7. [12 marks] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\begin{aligned}
\det(\lambda I - A) &= \det \begin{bmatrix} \lambda & -1 & 1 \\ -1 & \lambda & -1 \\ 1 & -1 & \lambda \end{bmatrix} = \det \begin{bmatrix} \lambda & -1 & 1 \\ -1 & \lambda & -1 \\ 0 & \lambda - 1 & \lambda - 1 \end{bmatrix} \\
&= (\lambda - 1) \det \begin{bmatrix} \lambda & -1 & 1 \\ -1 & \lambda & -1 \\ 0 & 1 & 1 \end{bmatrix} = (\lambda - 1) \det \begin{bmatrix} \lambda & -2 & 1 \\ -1 & \lambda + 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= (\lambda - 1)\,(\lambda(\lambda + 1) - 2) = (\lambda - 1)(\lambda^2 + \lambda - 2) \\
&= (\lambda - 1)^2(\lambda + 2)
\end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -2$.

Step 2: Find mutually orthogonal eigenvectors of A.

$$E_{-2}(A) = \text{null} \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$$E_1(A) = \text{null} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

OR: use either $E_1(A) = (E_{-2}(A))^\perp$ or $E_{-2}(A) = (E_1(A))^\perp$ to simplify calculations.

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

8. [12 marks] Let $X_1 = [\ 1\ \ 3\ \ 1\ \ 0\ ]^T, X_2 = [\ 2\ \ 4\ \ 1\ \ -1\ ]^T, X_3 = [\ 1\ \ 5\ \ 0\ \ 2\ ]^T$.

(a) [6 marks] Show that $S = \{X_1, X_2, X_3\}$ is linearly independent.

Solution: Let $a_1X_1 + a_2X_2 + a_3X_3 = O$. Then the augmented matrix for the system (with variables a_1, a_2, a_3) is

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 3 & 4 & 5 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

whence $a_1 = a_2 = a_3 = 0$, and S is linearly independent.

(b) [6 marks] Show that $U = \{X \in \mathbb{R}^4 \mid \det[\ X_1 \mid X_2 \mid X_3 \mid X\ ] = 0\}$ is a subspace of $\mathbb{R}^4$ and find a basis for U.

Solution: show U is one of the types of sets we know is a subspace.
Method 1:

$$X \in U \Rightarrow \det[\ X_1 \mid X_2 \mid X_3 \mid X\ ] = 0 \Rightarrow$$

$$\{X_1, X_2, X_3, X\} \text{ is linearly dependent} \Rightarrow X \in \operatorname{span}\{X_1, X_2, X_3\}, \text{ by part (a)}$$

On the other hand,

$$X \in \operatorname{span}\{X_1, X_2, X_3\} \Rightarrow \det[\ X_1 \mid X_2 \mid X_3 \mid X\ ] = 0 \Rightarrow X \in U.$$

Thus $U = \operatorname{span}\{X_1, X_2, X_3\}$ and $S = \{X_1, X_2, X_3\}$ is a basis for U.
Method 2: use the cofactor expansion of $[\ X_1 \mid X_2 \mid X_3 \mid X\ ]$ along column 4 to obtain

$$\det[\ X_1 \mid X_2 \mid X_3 \mid X\ ] = x_1C_{41} + x_2C_{42} + x_3C_{43} + x_4C_{44}.$$

Thus $U = \operatorname{null}[\ C_{41}\ \ C_{42}\ \ C_{43}\ \ C_{44}\ ]$. Moreover $X_i \in U$ since

$$\det[\ X_1 \mid X_2 \mid X_3 \mid X_i\ ] = 0,$$

for $i = 1, 2, 3$. Since $\dim(U) = 3$ and S is a set of 3 linearly independent vectors in U, S is a basis for U.
Method 3: $C = [\ C_{41}\ \ C_{42}\ \ C_{43}\ \ C_{44}\ ]^T$, as in Method 2. Then $U = (\operatorname{span}\{C\})^\perp$ and $\dim U = 4 - 1 = 3$. Then as in Method 2, S is a basis for U.
Method 4: use the subspace test directly, but first re-write U as
$U = \{X \in \mathbb{R}^4 \mid 7x_1 - 3x_2 + 2x_3 + 4x_4 = 0\}$, where we simplified C from above, or equivalently, calculated the determinant. Then it is straightforward to show that
U is non-empty:
U is closed under scalar multiplication:
U is closed under vector addition: