University of Toronto

FACULTY OF APPLIED SCIENCE AND ENGINEERING

Solutions to **FINAL EXAMINATION, DECEMBER, 2013**

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - LINEAR ALGEBRA

Exam Type: A

General Comments:

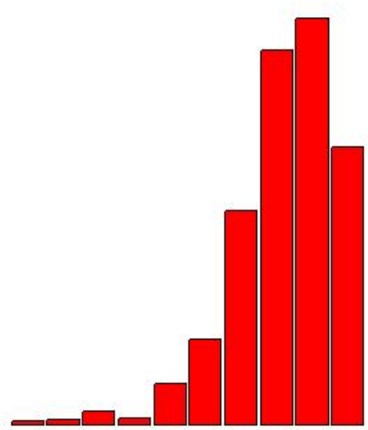
1. The results on this exam were very good. This was probably due to the routine nature of all the problems on this exam. The intent was to compensate for the poor results on the midterm test, and we seem to have been successful!

2. In Question 5 many students confused the notions of basis, spans and subspaces. So for instance if the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for null(A), then it is incorrect to say null$(A) = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ or a basis is span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It is correct to say that null$(A) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, but you have to identify what the basis is to get full marks.

3. In Question 7 many students let
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
and then proceeded to set up a system of nine equations in nine variables. This can work but it is not the easiest way to do it! Moreover, some students then used their calculators to solve their system(s), which would cost you 4 marks. Recall, as stated on the course information sheet: Use of a calculator will be permitted during all quizzes, tests and exams. However, it is still your responsibility to explain your work. A correct answer with no justification will receive no marks.

Breakdown of Results: 865 students wrote this exam. The marks ranged from 3.3% to 100%, and the average was 76.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	19.4%
A	47.8%	80-89%	28.4%
B	26.2%	70-79%	26.2%
C	15.0%	60-69%	15.0%
D	6.0%	50-59%	6.0%
F	5.0%	40-49%	3.0%
		30-39%	0.5%
		20-29%	1.0%
		10-19%	0.3%
		0-9%	0.2%



1. [10 marks] The parts of this question are unrelated.

(a) [5 marks] Calculate $\det\begin{bmatrix} 2 & 0 & -1 & 5 \\ -1 & 3 & 0 & 1 \\ 5 & 4 & 2 & 3 \\ 1 & 6 & 0 & 4 \end{bmatrix}$.

Solution: many possible ways to calculate this. Here's one:

$$\begin{aligned}
\det\begin{bmatrix} 2 & 0 & -1 & 5 \\ -1 & 3 & 0 & 1 \\ 5 & 4 & 2 & 3 \\ 1 & 6 & 0 & 4 \end{bmatrix} &= \det\begin{bmatrix} 2 & 0 & -1 & 5 \\ -1 & 3 & 0 & 1 \\ 9 & 4 & 0 & 13 \\ 1 & 6 & 0 & 4 \end{bmatrix} \\
&= (-1)\det\begin{bmatrix} -1 & 3 & 1 \\ 9 & 4 & 13 \\ 1 & 6 & 4 \end{bmatrix} \\
&= -\det\begin{bmatrix} -1 & 3 & 1 \\ 0 & 31 & 22 \\ 0 & 9 & 5 \end{bmatrix} \\
&= +\det\begin{bmatrix} 31 & 22 \\ 9 & 5 \end{bmatrix} \\
&= (155 - 198) \\
&= -43
\end{aligned}$$

(b) [5 marks] Find a basis for $S^\perp$ if $S = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 3 \end{bmatrix}\right\}$.

Solution: $\dim(S^\perp) = 4 - \dim(S) = 4 - 2 = 2$, so any two vectors that span $S^\perp$ will form a basis for $S^\perp$, as would any two independent vectors in $S^\perp$. We have

$$S^\perp = \operatorname{null}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 5 & 3 \end{bmatrix} = \left\{\begin{bmatrix} s-2t \\ -5s-3t \\ s \\ t \end{bmatrix} \middle| s, t \in \mathbb{R}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

So a basis for $S^\perp$ is

$$\left\{\begin{bmatrix} 1 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

2. [10 marks]

(a) [5 marks] Find all values of c for which $\det \begin{bmatrix} 1 & c & c \\ 1 & -1 & 1 \\ c & c & 1 \end{bmatrix} = 0$.

Solution:

$$\det \begin{bmatrix} 1 & c & c \\ 1 & -1 & 1 \\ c & c & 1 \end{bmatrix} = -1 + c^2 + c^2 + c^2 - c - c = 3c^2 - 2c - 1 = (3c+1)(c-1)$$

So

$$\det \begin{bmatrix} 1 & c & c \\ 1 & -1 & 1 \\ c & c & 1 \end{bmatrix} = 0 \Leftrightarrow (3c+1)(c-1) = 0 \Leftrightarrow c = 1 \text{ or } c = -\frac{1}{3}.$$

(b) [5 marks] Find all values of c for which the system of equations

$$\begin{array}{rcrcrcl} x_1 & + & cx_2 & + & cx_3 & = & 2 \\ x_1 & - & x_2 & + & x_3 & = & 4 \\ cx_1 & + & cx_2 & + & x_3 & = & 2 \end{array}$$

has *(i)* no solution, *(ii)* a unique solution, *(iii)* infinitely many solutions.

Solution: note the coefficient matrix of this system is the matrix from part (a) for which we know the coefficient matrix is invertible if and only if $c \neq 1, c \neq -1/3$. Therefore, *(ii)*, the system has a unique solution if $c \neq 1, c \neq -1/3$.
Now see what happens for $c = 1$ and $c = -1/3$:
(iii): if $c = 1$ the system becomes

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 2 \\ x_1 & - & x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & 2 \end{array}$$

which has infinitely many solutions since the first and third equations are the same: it reduces to a system of three equations in two variables.
(i): if $c = -1/3$ the system has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 2 \\ 1 & -1 & 1 & 4 \\ -1/3 & -1/3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 2 \\ 0 & -2/3 & 4/3 & 2 \\ 0 & -4/3 & 8/3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 2 \\ 0 & -2/3 & 4/3 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right],$$

so there is no solution to the system in this case.

3. [10 marks; 2 marks each.] Indicate if the following statements are True or False, and give a *brief* explanation why. Circle your choice.

(a) If λ is an eigenvalue of the matrix A then λ^2 is an eigenvalue of the matrix A^2.

True **False**

True: let a corresponding eigenvector be $\mathbf{x}$; so $\mathbf{x}$ is non-zero and $A\mathbf{x} = \lambda\mathbf{x}$. Then $A^2\,\mathbf{x} = A(A\,\mathbf{x}) = A(\lambda\,\mathbf{x}) = \lambda(A\,\mathbf{x}) = \lambda(\lambda\,\mathbf{x}) = \lambda^2\,\mathbf{x}$. So λ^2 is an eigenvalue of A^2.

(b) If A is an $n \times n$ diagonalizable matrix then $A + 3I_n$ is also diagonalizable.

True **False**

True: there is an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$. Then $P^{-1}(A + 3I_n)P = P^{-1}AP + 3P^{-1}P = D + 3I_n$.
This last matrix is also a diagonal matrix, so $A + 3I_n$ is diagonalizable.

(c) If the rows of the 5×7 matrix A are linearly independent then $\text{null}(A) = \{\mathbf{0}\}$.

True **False**

False: the rank of A is 5 so $\dim(\text{null}(A)) = 7 - 5 = 2 \neq 0$. Hence $\text{null}(A) \neq \{\mathbf{0}\}$.
Aside: if the *columns* of A are independent, it is true, but seven columns can't be independent if there are only five rows.

(d) If $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is a linear transformation and T is one-to-one, then T is onto.

True: if T is one-to-one then $\dim(\text{kernel}(T)) = 0$, so $\dim(\text{range}(T)) = 4 - 0 = 4$, which means $\text{range}(T) = \mathbb{R}^4$, so T is onto.
Note: this result is included in The Big Theorem.

(e) If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal column vectors in $\mathbb{R}^n$ then $\mathbf{y}\,\mathbf{x}^T = 0$.

True **False**

False: if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal column vectors then $\mathbf{x}^T\mathbf{y} = 0$, but $\mathbf{x}\mathbf{y}^T$ is the $n \times n$ matrix $[x_i y_j]$, for $1 \le i, j \le n$, which need not be zero. For example in $\mathbb{R}^2$,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{x} \cdot \mathbf{y} = 0, \text{ but } \mathbf{x}\mathbf{y}^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

4. [10 marks] Find the solution to the system of linear differential equations

$$\begin{array}{rcrcr} y_1' & = & 3y_1 & - & 2y_2 \\ y_2' & = & -4y_1 & + & y_2 \end{array}$$

where y_1, y_2 are functions of t, and $y_1(0) = 4, y_2(0) = -1$.

Solution: let the coefficient matrix be $A = \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$. We need the eigenvalues and eigenvectors of A.

$$\det(\lambda I - A) = (\lambda - 3)(\lambda - 1) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 5.$$

For the eigenvalue $\lambda_1 = -1$:

$$(\lambda_1 I_2 - A|\mathbf{0}) = \left[\begin{array}{rr|r} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]; \text{ so take } \mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = 5$:

$$(\lambda_2 I_2 - A|\mathbf{0}) = \left[\begin{array}{rr|r} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]; \text{ so take } \mathbf{u_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then the general solution to the system of differential equations is

$$\mathbf{y} = c_1 \mathbf{u}_1 e^{-t} + c_2 \mathbf{u}_2 e^{5t}.$$

To find c_1, c_2 use the initial conditions, with $t = 0$:

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} - 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t} \Rightarrow y_1 = e^{-t} + 3e^{5t};\ y_2 = 2e^{-t} - 3e^{5t}.$$

5. [10 marks] The reduced echelon form of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -2 & 0 & 9 & -16 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [1 mark] The rank of A is $\underline{r = 2\text{, the number of pivot entries in } R}$.

(b) [2 marks] $\dim(\text{row}(A)) = \underline{r = 2}$ and a basis for the row space of A is:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 9 \\ -16 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 5 \end{bmatrix} \right\},$$

the non-zero rows of R; or any two rows of A.

(c) [3 marks] $\dim(\text{col}(A)) = \underline{r = 2}$ and a basis for the column space of A is:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\},$$

the first and third columns of A; or any two non-parallel columns of A.

(d) [4 marks] $\dim(\text{null}(A)) = \underline{5 \text{ - r} = 3}$ and a basis for the null space of A is:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\},$$

(or any other three independent solutions to $R\mathbf{x} = \mathbf{0}$) since the general solution to $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 9t + 16u \\ s \\ 3t - 5u \\ t \\ u \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix},$$

for parameters s, t, u.

6. [10 marks] Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ 2x + y \end{bmatrix}.$$

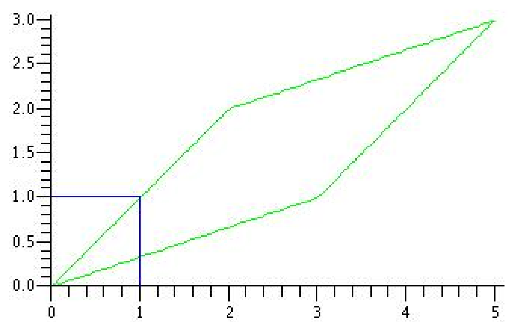
(a) [5 marks] Draw the image of the unit square under T, and calculate its area.

Solution: the image of the unit square is the parallelogram determined by

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

In the diagram to the right, the unit square is in blue, and the image of the unit square is in green. Its area is

$$\left|\det \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\right| = |2 - 6| = 4.$$



(b) [5 marks] Find the formula for $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

Solution: if the matrix of T is

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

then the matrix of T^{-1} is

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix},$$

so the formula for T^{-1} is

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{4} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -x + 3y \\ 2x - 2y \end{bmatrix}.$$

7. [10 marks] Suppose $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},\ T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = -2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = -\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Find a 3×3 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$, for all $\mathbf{x}$ in $\mathbb{R}^3$.

Solution: let A be the matrix of T so that $T(\mathbf{x}) = A\mathbf{x}$. We are given

$$A\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, A\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Method 1: the three given matrix equations can be combined into one matrix equation

$$A\left[\begin{array}{r|r|r} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{array}\right] = \left[\begin{array}{r|r|r} 3 & -2 & -1 \\ 6 & -2 & 0 \\ -3 & -2 & -2 \end{array}\right] \Leftrightarrow A = \begin{bmatrix} 3 & -2 & -1 \\ 6 & -2 & 0 \\ -3 & -2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}^{-1}$$

To find the inverse use the Gaussian algorithm or the adjoint formula. Here are the calculations for the latter method:

$$\frac{1}{2+2+1-4}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \frac{1}{1}\begin{bmatrix} 2 & -4 & 3 \\ -1 & 3 & -2 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & -1 \\ -4 & 3 & 2 \\ 3 & -2 & -1 \end{bmatrix}.$$

Thus

$$A = \begin{bmatrix} 3 & -2 & -1 \\ 6 & -2 & 0 \\ -3 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -4 & 3 & 2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 11 & -7 & -6 \\ 20 & -12 & -10 \\ -4 & 1 & 1 \end{bmatrix}.$$

Method 2: we have three eigenvectors of A corresponding to distinct eigenvalues. Let

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then A is diagonalizable and

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}^{-1}$$

So

$$A = (PD)P^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ 6 & -2 & 0 \\ -3 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -4 & 3 & 2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 11 & -7 & -6 \\ 20 & -12 & -10 \\ -4 & 1 & 1 \end{bmatrix}.$$

as in Method 1.

8. [10 marks] Let $S = \text{span}\left\{[\begin{array}{cccc}1 & 0 & 0 & 1\end{array}]^T, [\begin{array}{cccc}0 & 1 & 0 & 1\end{array}]^T, [\begin{array}{cccc}1 & 1 & 1 & 1\end{array}]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

Solution: call the three given vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and apply the Gram-Schmidt algorithm to find an orthogonal basis $\mathbf{f}_2, \mathbf{f}_2, \mathbf{f}_3$. Take $\mathbf{f}_1 = \mathbf{x}_1$,

$$\mathbf{f}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{f}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

Optional: clear fractions and take

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{f}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{f}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

Either way $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \left[\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\right]^T$. Find $\text{proj}_S(\mathbf{x})$.

Solution: using $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ with fractions cleared.

$$\text{proj}_S\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 = \frac{5}{2}\mathbf{f}_1 + \frac{7}{6}\mathbf{f}_2 + \frac{8}{12}\mathbf{f}_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

Cross-check/Alternate Solution: $S^\perp = \text{span}\{\mathbf{y}\}$ with $\mathbf{y} = [\begin{array}{cccc}1 & 1 & -1 & -1\end{array}]^T$. Then

$$\text{proj}_S\mathbf{x} = \mathbf{x} - \text{proj}_{S^\perp}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} = \mathbf{x} + \frac{4}{4}\mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

9. [10 marks] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda \end{bmatrix} = (\lambda - 1) \det \begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} \\ &= (\lambda - 1)\left(\lambda^2 - 1\right) = (\lambda - 1)^2(\lambda + 1) \end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 1$, repeated, and $\lambda_2 = -1$.

Step 2: Find three mutually orthogonal eigenvectors of A. Notation: let

$$E_\lambda(A) = \{\mathbf{x} \in \mathbb{R}^3 \mid A\,\mathbf{x} = \lambda\,\mathbf{x}\}$$

be the eigenspace of A corresponding to the eigenvalue λ.

$$E_1(A) = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$E_{-1}(A) = \text{null} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

OR: use either $E_1(A) = (E_{-1}(A))^\perp$ or $E_{-1}(A) = (E_1(A))^\perp$ to simplify calculations. That is, $E_1(A)$ is the plane with equation $x - z = 0$ and $E_{-1}(A)$ is the line normal to the plane.

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}; \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$