UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

SOLUTIONS TO FINAL EXAMINATION, DECEMBER 2015

DURATION: 2 AND 1/2 HRS

FIRST YEAR - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - Linear Algebra

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Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

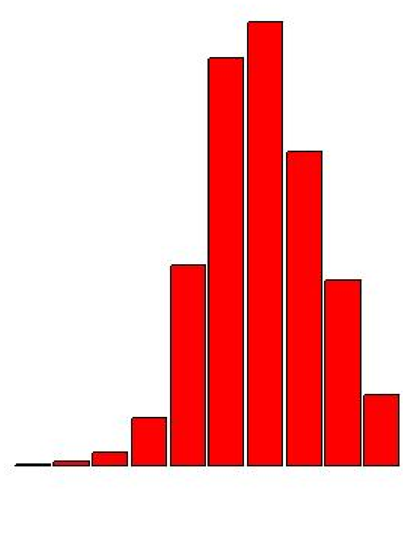
1. Questions 3, 4 and 8 were the only questions with a failing average; the other questions were well done. Question 3 was supposed to be challenging—and only two students got it completely correct.

2. Question 4 was *not* supposed to be challenging, but it turned out that while 20 students got it completely correct, 414 students got 0 or 1 on it! Apparently many students had no real idea of what the question was about.

3. 92 students got Question 8 completely perfect; part (a) was based on #61 in Sec 6.1.

Breakdown of Results: 925 students wrote this exam. The marks ranged from 0.0% to 96.25%, and the average was 63.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.2%
A	15.2%	80-89%	11.0%
B	18.6%	70-79%	18.6%
C	26.3%	60-69%	26.3%
D	24.1%	50-59%	24.1%
F	15.8%	40-49%	11.9%
		30-39%	2.8%
		20-29%	0.8%
		10-19%	0.2%
		0-9%	0.1%



1. [avg: 8.7/10] Let

$$\mathbf{x}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\mathbf{u}_1=\begin{bmatrix}1\\0\\1\\-1\end{bmatrix},\mathbf{u}_2=\begin{bmatrix}1\\1\\-1\\0\end{bmatrix},\mathbf{u}_3=\begin{bmatrix}3\\-1\\2\\5\end{bmatrix},\mathbf{u}_4=\begin{bmatrix}-2\\5\\3\\1\end{bmatrix}.$$

Show $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$ is an orthogonal set, and write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$ and $\mathbf{u}_4$.

Solution: need to show $\mathbf{u}_i\cdot\mathbf{u}_j=0$, for $i\neq j$. Since $\mathbf{u}_i\cdot\mathbf{u}_j=\mathbf{u}_j\cdot\mathbf{u}_i$, there are only six dot products you must check:

$$\mathbf{u}_1\cdot\mathbf{u}_2=1+0-1+0=0,\quad \mathbf{u}_1\cdot\mathbf{u}_3=3+0+2-5=0,\quad \mathbf{u}_1\cdot\mathbf{u}_4=-2+0+3-1=0,$$

$$\mathbf{u}_2\cdot\mathbf{u}_3=3-1-2+0=0,\quad \mathbf{u}_2\cdot\mathbf{u}_4=-2+5-3+0=0,\quad \mathbf{u}_3\cdot\mathbf{u}_4=-6-5+6+5=0.$$

Thus $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$ is an independent set of four vectors in $\mathbf{R}^4$, so it is a basis for $\mathbf{R}^4$, and

$$\begin{aligned}\mathbf{x}&=\frac{\mathbf{x}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1+\frac{\mathbf{x}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2+\frac{\mathbf{x}\cdot\mathbf{u}_3}{\|\mathbf{u}_3\|^2}\mathbf{u}_3+\frac{\mathbf{x}\cdot\mathbf{u}_4}{\|\mathbf{u}_4\|^2}\mathbf{u}_4\\&=\frac{1+3-4}{1+1+2}\mathbf{u}_1+\frac{1+2-3}{1+1+1}\mathbf{u}_2+\frac{3-2+6+20}{9+1+4+25}\mathbf{u}_3+\frac{-2+10+9+4}{4+25+9+1}\mathbf{u}_4\\&=0\,\mathbf{u}_1+0\,\mathbf{u}_2+\frac{9}{13}\mathbf{u}_3+\frac{7}{13}\mathbf{u}_4\\&=\frac{9}{13}\mathbf{u}_3+\frac{7}{13}\mathbf{u}_4\end{aligned}$$

OR, do it the long way: solve the vector equation $\mathbf{x}=x_1\,\mathbf{u}_1+x_2\,\mathbf{u}_2+x_3\,\mathbf{u}_3+x_4\,\mathbf{u}_4$ for x_1,x_2,x_3,x_4 by solving a linear system of equations. Using augmented matrices and good old row reduction:

$$\left[\begin{array}{rrrr|r}1&1&3&-2&1\\0&1&-1&5&2\\1&-1&2&3&3\\-1&0&5&1&4\end{array}\right]\sim\left[\begin{array}{rrrr|r}1&0&4&-7&-1\\0&1&-1&5&2\\0&-1&-2&10&4\\0&0&9&-6&3\end{array}\right]\sim\left[\begin{array}{rrrr|r}1&0&4&-7&-1\\0&1&-1&5&2\\0&0&-3&15&6\\0&0&3&-2&1\end{array}\right]$$

$$\sim\left[\begin{array}{rrrr|r}1&0&0&13&7\\0&1&0&0&0\\0&0&1&-5&-2\\0&0&0&13&7\end{array}\right]\sim\left[\begin{array}{rrrr|r}1&0&0&0&0\\0&1&0&0&0\\0&0&13&0&9\\0&0&0&13&7\end{array}\right],$$

from which

$$x_1=0,\ x_2=0,\ x_3=\frac{9}{13},\ x_4=\frac{7}{13},$$

as before.

Continued...

2. [avg: 7.8/10] Let $A = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$.

Find the eigenvalues of A and a basis for each eigenspace of A. Plot the eigenspaces of A in $\mathbf{R}^2$, and clearly indicate which eigenspace corresponds to which eigenvalue.

Solution: be careful with the fractions!

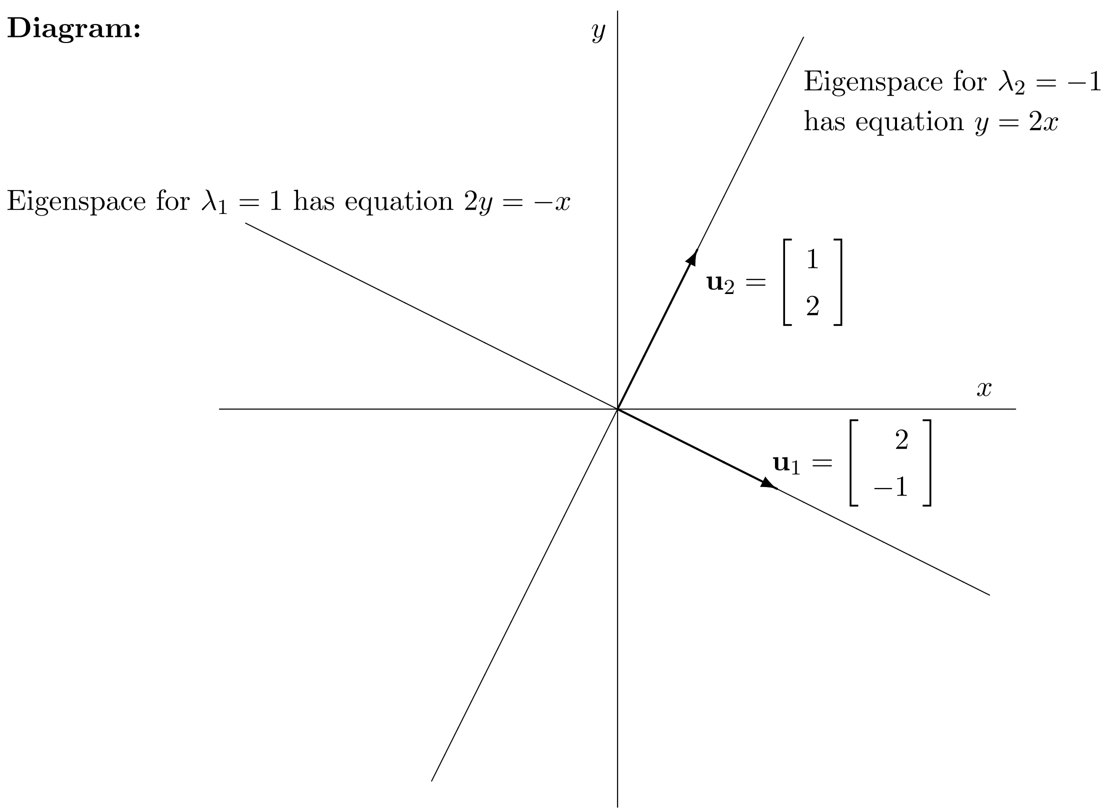
$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda - 3/5 & 4/5 \\ 4/5 & \lambda + 3/5 \end{bmatrix} = \lambda^2 - \frac{9}{25} - \frac{16}{25} = \lambda^2 - 1,$$

so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. Next, find a basis for each eigenspace:

$$\text{null}\begin{bmatrix} \lambda_1 - 3/5 & 4/5 \\ 4/5 & \lambda_1 + 3/5 \end{bmatrix} = \text{null}\begin{bmatrix} 2/5 & 4/5 \\ 4/5 & 8/5 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\};$$

$$\text{null}\begin{bmatrix} \lambda_2 - 3/5 & 4/5 \\ 4/5 & \lambda_2 + 3/5 \end{bmatrix} = \text{null}\begin{bmatrix} -8/5 & 4/5 \\ 4/5 & -2/5 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}.$$

Diagram:



The eigenvectors are orthogonal to each other; each eigenspace is a line passing through the origin.

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3.[avg: 3.3/10] An $n \times n$ matrix A is called **idempotent** if $A^2 = A$.

(a) [2 marks] Show that the only invertible idempotent matrix is the identity matrix.

Solution: if A is invertible then $A^2 = A \Rightarrow A^{-1}A^2 = A^{-1}A \Rightarrow A = I$.

(b) [2 marks] Show that if A is idempotent then so is $I - A$, where I is the $n \times n$ identity matrix.

Solution: $(I - A)^2 = (I - A)(I - A) = \underbrace{I - 2\,A + A^2 = I - 2\,A + A}_{\boxed{\text{since } A^2 = A}} = I - A$

(c) [3 marks] Let $\mathbf{v}$ be an eigenvector of an idempotent matrix A, with corresponding eigenvalue λ. Show that $\lambda = 0$ or $\lambda = 1$.

Solution: start with the definition of eigenvector and eigenvalue:

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ \Rightarrow A^2\,\mathbf{v} &= A(\lambda\,\mathbf{v}) \\ (\text{using } A^2 = A) \Rightarrow A\,\mathbf{v} &= \lambda\,A\,\mathbf{v} \\ \Rightarrow \lambda\,\mathbf{v} &= \lambda(\lambda\,\mathbf{v}) = \lambda^2\mathbf{v} \\ \Rightarrow \lambda &= \lambda^2, \text{ snce } \mathbf{v} \text{ is non-zero} \\ \Rightarrow \lambda &= 0 \text{ or } 1 \end{aligned}$$

(d) [3 marks] Show that every idempotent matrix is diagonalizable. Hint: what can you say about the dimension of the eigenspaces of A?

Solution: Recall: the eigenspace of A corresponding to the eigenvalue λ is null$(A - \lambda\,I)$.

$$\begin{aligned} A^2 = A \quad &\Rightarrow \quad A^2 - A = 0 \\ &\Rightarrow \quad A(A - I) = 0 \\ &\Rightarrow \quad \text{col}(A - I) \text{ is contained in null}(A) \\ &\Rightarrow \quad \dim(\text{col}(A - I)) \le \dim(\text{null}(A)) \\ (\text{by rank-nullity theorem}) \quad &\Rightarrow \quad n - \dim(\text{null}(A - I)) \le \dim(\text{null}(A)) \\ &\Rightarrow \quad n \le \dim(\text{null}(A)) + \dim(\text{null}(A - I)) \end{aligned}$$

Since eigenvectors corresponding to distinct eigenvalues are automatically independent, A has n independent eigenvectors, which means A is diagonalizable.

4. [avg: 2.7/10] Find *all* linear transformations $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ such that

$$\ker(T) = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ and } \text{range}(T) = \text{span}\left\{ \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \right\}.$$

Solution 1: let the matrix of T be A. Then

$$\text{range}(T) = \text{col}(A) = \text{span}\left\{ \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \right\} \Rightarrow A = \begin{bmatrix} 4a & 4b & 4c \\ 3a & 3b & 3c \\ 7a & 7b & 7c \end{bmatrix}$$

for some scalars a, b, c. Since $\ker(T) = \text{null}(A)$, we have

$$\begin{bmatrix} 4a & 4b & 4c \\ 3a & 3b & 3c \\ 7a & 7b & 7c \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 4a & 4b & 4c \\ 3a & 3b & 3c \\ 7a & 7b & 7c \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \left\{ \begin{array}{rcrcrcl} 2a & - & b & + & 2c & = & 0 \\ 2a & + & 2b & - & c & = & 0 \end{array} \right. ,$$

which has solution $b = -2a,\ c = -2a$, for a a parameter. Thus

$$A = \begin{bmatrix} 4a & -8a & -8a \\ 3a & -6a & -6a \\ 7a & -14a & -14a \end{bmatrix} = a \begin{bmatrix} 4 & -8 & -8 \\ 3 & -6 & -6 \\ 7 & -14 & -14 \end{bmatrix}.$$

for any scalar $a \neq 0$; and $T(\mathbf{x}) = A\,\mathbf{x}$.

Solution 2: (For more solutions to this question, see Page 10.) Use the fact that $\text{row}(A) = (\text{null}(A))^{\perp}$. Then

$$\text{row}(A) = \text{null}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

Thus each row of A is a multiple of $\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}$ and each column of A is a multiple of $\begin{bmatrix} 4 & 3 & 7 \end{bmatrix}^T$. Consequently

$$A = a \begin{bmatrix} 4 & -8 & -8 \\ 3 & -6 & -6 \\ 7 & -14 & -14 \end{bmatrix}.$$

5. [avg: 8.2/10] Find the solution to the system of linear differential equations $\left\{\begin{array}{rcrcr} y_1' & = & y_1 & + & 3y_2 \\ y_2' & = & 2y_1 & + & 2y_2 \end{array}\right.$, where y_1, y_2 are functions of t, and $y_1(0) = 0,\ y_2(0) = 5$.

Solution: let the coefficient matrix be $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 2) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0 \Rightarrow \lambda = -1 \text{ or } 4.$$

For the eigenvalue $\lambda_1 = -1$:

$$(\lambda_1 I - A|\mathbf{0}) = \left[\begin{array}{rr|r} -2 & -3 & 0 \\ -2 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{so take } \mathbf{u_1} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = 4$:

$$(\lambda_2 I - A|\mathbf{0}) = \left[\begin{array}{rr|r} 3 & -3 & 0 \\ -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{so take } \mathbf{u_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then the general solution to the system of differential equations is

$$\mathbf{y} = c_1 \mathbf{u}_1 e^{-t} + c_2 \mathbf{u}_2 e^{4t}.$$

To find the constants c_1, c_2, use the initial conditions, with $t = 0$:

$$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = - \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$

and

$$y_1 = -3\, e^{-t} + 3\, e^{4t};\ y_2 = 2\, e^{-t} + 3\, e^{4t}.$$

Continued...

6. [avg: 7.1/10] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\begin{aligned}
\det \begin{bmatrix} \lambda-8 & 2 & -2 \\ 2 & \lambda-5 & -4 \\ -2 & -4 & \lambda-5 \end{bmatrix} &= \det \begin{bmatrix} \lambda-8 & 2 & -2 \\ 2 & \lambda-5 & -4 \\ 0 & \lambda-9 & \lambda-9 \end{bmatrix} = (\lambda-9)\det \begin{bmatrix} \lambda-8 & 2 & -2 \\ 2 & \lambda-5 & -4 \\ 0 & 1 & 1 \end{bmatrix} \\
&= (\lambda-9)\det \begin{bmatrix} \lambda-8 & 4 & -2 \\ 2 & \lambda-1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = (\lambda-9)\det \begin{bmatrix} \lambda-8 & 4 \\ 2 & \lambda-1 \end{bmatrix} \\
&= (\lambda-9)(\lambda^2-9\lambda+8-8) = (\lambda-9)(\lambda-9)\lambda = \lambda\,(\lambda-9)^2
\end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 9$, repeated.

Step 2: recall $\text{null}(A - \lambda I)$ is the eigenspace of A corresponding to the eigenvalue λ.
Find three mutually **orthogonal** eigenvectors of A. For $\lambda_1 = 0$, $\text{null}(A) =$

$$\text{null}\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix} = \text{null}\begin{bmatrix} 4 & -1 & 1 \\ 0 & 18 & 18 \\ 0 & 18 & 18 \end{bmatrix} = \text{null}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right\},$$

$$\text{and for } \lambda_2 = 9,\ \text{null}(9\,I - A) = \text{null}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}\right\}.$$

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} 1/3 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{bmatrix};\quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Continued...

7. [avg: 8.2/10] Let $S = \text{span}\left\{[\,1 \ \ -1 \ \ 0 \ \ 0\,]^T, [\,1 \ \ 0 \ \ 1 \ \ 0\,]^T, [\,0 \ \ 1 \ \ 1 \ \ 1\,]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

Solution: call the three given vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and apply the Gram-Schmidt algorithm to find an orthogonal basis $\mathbf{f}_2, \mathbf{f}_2, \mathbf{f}_3$. Take $\mathbf{f}_1 = \mathbf{x}_1$,

$$\mathbf{f}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix},$$

$$\mathbf{f}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 2 & 0 & 3 & 1 \end{bmatrix}^T$. Find $\text{proj}_S(\mathbf{x})$.

Solution: use the projection formula, and your orthogonal basis from part (a).

$$\text{proj}_S\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ 1 \\ 8 \\ 3 \end{bmatrix}.$$

Alternate Solution: observe that $S^\perp = \text{span}\{\mathbf{y}\}$ with $\mathbf{y} = [\,1 \ \ 1 \ \ -1 \ \ 0\,]^T$. Then

$$\text{proj}_S\mathbf{x} = \mathbf{x} - \text{proj}_{S^\perp}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} = \mathbf{x} + \frac{1}{3}\mathbf{y} = \frac{1}{3}[\,7 \ \ 1 \ \ 8 \ \ 3\,]^T.$$

8. [avg: 4.8/10] Let $A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$.

(a) [4 marks] Show that if $a + b + c = 0$, then A is not invertible.

Solution: adding the second and third rows of A to the first row of A gives

$$\det(A) = \det \begin{bmatrix} a+b+c & c+a+b & b+c+a \\ b & a & c \\ c & b & a \end{bmatrix} = \det \begin{bmatrix} 0 & 0 & 0 \\ b & a & c \\ c & b & a \end{bmatrix} = 0.$$

Since $\det(A) = 0$, A is not invertible.

Alternate Solution: observe that

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+c+b \\ b+a+c \\ c+b+a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This means that the null space of A is not the trivial subspace, so A is not invertible.

Alternate Solution: if you calculate the determinant of A directly, and factor it, you obtain

$$\det(A) = a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2 - ab - bc - ca).$$

But the factorization is hardly obvious! Then: $a + b + c = 0 \Rightarrow \det(A) = 0$, and so A is not invertible.

(b) [6 marks] Show that if $a+b+c = \pm 1$ and $a^2+b^2+c^2 = 1$, then A is orthogonal. Hint: $(\pm 1)^2 = 1$.

Solution: use the hint:

$$(a+b+c)^2 = 1 \Leftrightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1.$$

Since it is given that $a^2 + b^2 + c^2 = 1$, it follows that

$$ab + bc + ca = 0.$$

That is, the columns of A (or the rows of A) form an orhonormal basis of $\mathbf{R}^3$, which means the matrix A is orthogonal.

Some alternate solutions:

Question 4, Solution 3 Outline: Observe that the two given basis vectors of $\ker(T)$ are orthogonal. Extend them to an orthonormal basis

$$\left\{ \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

of $\mathbf{R}^3$. Then if A is the matrix of T, we need

$$A\left(\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 4k \\ 0 & 0 & 3k \\ 0 & 0 & 7k \end{bmatrix}, \text{ since } T\left(\frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}\right) \text{ is in range}(T),$$

$$\Rightarrow A = \begin{bmatrix} 0 & 0 & 4k \\ 0 & 0 & 3k \\ 0 & 0 & 7k \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}\right)^T = \frac{k}{3}\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix} = \frac{k}{3}\begin{bmatrix} -4 & 8 & 8 \\ -3 & 6 & 6 \\ -7 & 14 & 14 \end{bmatrix},$$

for some scalar k.

Question 4, Solution 4 Outline: If A is the matrix of T, then A must be diagonalizable, since

$$A\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \; A\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \; A\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = k\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}.$$

Then $D = P^{-1}AP \Leftrightarrow A = PDP^{-1}$. Calculating gives

$$\begin{aligned} A &= \begin{bmatrix} 2 & 2 & 4 \\ -1 & 2 & 3 \\ 2 & -1 & 7 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ -1 & 2 & 3 \\ 2 & -1 & 7 \end{bmatrix}^{-1} \\ \text{(have to find the inverse)} &= \begin{bmatrix} 0 & 0 & 4k \\ 0 & 0 & 3k \\ 0 & 0 & 7k \end{bmatrix}\frac{1}{48}\begin{bmatrix} 17 & -18 & -2 \\ 13 & 6 & -10 \\ -3 & 6 & 6 \end{bmatrix} \\ &= \frac{k}{16}\begin{bmatrix} -4 & 8 & 8 \\ -3 & 6 & 6 \\ -7 & 14 & 14 \end{bmatrix} \end{aligned}$$

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