University of Toronto

Faculty of Applied Science and Engineering

Solutions to Final Examination, December 2017

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - Linear Algebra

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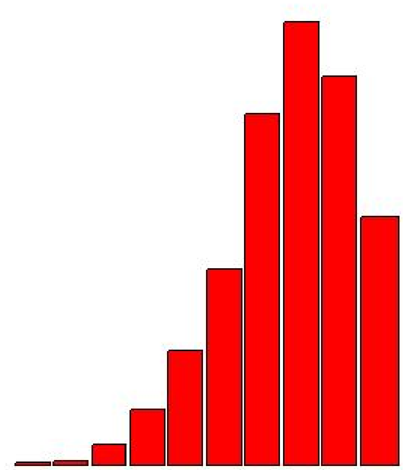
Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

- The only questions with failing averages were Questions 4 and 7. In Question 4, part (c), the second choice was done much better than the first choice. In Question 7, the common mistakes were (*i*) thinking the given basis was an orthogonal basis of S, and (*ii*) finding an orthogonal basis of S and then claiming it was a basis for $S^\perp$. But $\dim(S^\perp)$ is only 1. Questions 1, 2 and 5 were very well done.
- In Question 6 many students failed to find orthonormal eigenvectors and hence, did not get an orthogonal matrix P.
- In Question 8, many students did not seem aware of the Normal Equations, and tried to find the least squares line from first principles, which is the hard way!

Breakdown of Results: 811 registered students wrote this test. The marks ranged from 8.75% to 100%, and the average was 71.6%. There were ten perfect papers. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	14.9%
A	35.9%	80-89%	21.0%
B	23.9%	70-79%	23.9%
C	19.0%	60-69%	19.0%
D	10.6%	50-59%	10.6%
F	10.6%	40-49%	6.2%
		30-39%	3.0%
		20-29%	1.1%
		10-19%	0.2%
		0-9%	0.1%



1. [avg: 9.38/10] Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$; let $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthogonal set. Show your work! Then write $\vec{x}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ and $\vec{u}_4$.

Solution: show that each pair of vectors $\vec{u}_i, \vec{u}_j, i \neq j$ is orthogonal. There are six dot products you must calculate:

1. $\vec{u}_1 \cdot \vec{u}_2 = 1 - 1 + 1 - 1 = 0$
2. $\vec{u}_1 \cdot \vec{u}_3 = -1 + 0 + 1 - 0 = 0$
3. $\vec{u}_1 \cdot \vec{u}_4 = 0 + 1 + 0 - 1 = 0$
4. $\vec{u}_2 \cdot \vec{u}_3 = -1 + 0 + 1 - 0 = 0$
5. $\vec{u}_2 \cdot \vec{u}_4 = 0 - 1 + 0 + 1 = 0$
6. $\vec{u}_3 \cdot \vec{u}_4 = 0 + 0 + 0 + 0 = 0$

Now let $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4$. Using the results from Chapter 7 we can solve for c_i directly:

$$c_i = \frac{\vec{x} \cdot \vec{u}_i}{\|\vec{u}_i\|^2}.$$

Thus

$$c_1 = \frac{1+2+3+4}{4} = \frac{5}{2};\ c_2 = \frac{1-2+3-4}{4} = -\frac{1}{2};\ c_3 = \frac{-1+3}{2} = 1;\ c_4 = \frac{2-4}{2} = -1.$$

OR you can solve a system of linear equations for c_i by reducing an augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & -1 & 0 & -1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 1 \\ 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & -2 & 1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 2 \end{array}\right];$$

and as before, $c_1 = \frac{5}{2},\ c_2 = -\frac{1}{2},\ c_3 = 1,\ c_4 = -1$. Either way

$$\vec{x} = \frac{5}{2}\vec{u}_1 - \frac{1}{2}\vec{u}_2 + \vec{u}_3 - \vec{u}_4.$$

2. [avg: 9.3/10]

2.(a) [6 marks] Find all values of a for which the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & a & a \\ a & 2 & 1 \end{bmatrix}$ is **not** invertible.

Solution: use the fact that a matrix A is not invertible if and only if $\det(A) = 0$. We have

$$\det \begin{bmatrix} 1 & 1 & 3 \\ 2 & a & a \\ a & 2 & 1 \end{bmatrix} = a + a^2 + 12 - 3a^2 - 2a - 2 = -2a^2 - a + 10 = -(a-2)(2a+5).$$

Then

$$\det(A) = 0 \Leftrightarrow (a-2)(2a+5) = 0 \Leftrightarrow a = 2 \text{ or } a = -\frac{5}{2},$$

and so A is not invertible if $a = 2$ or $a = -5/2$.

2.(b) [4 marks] Let $A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 2 & 2 & 2 & 2 \\ 5 & 1 & 0 & 2 \\ 4 & -2 & 3 & 3 \end{bmatrix}$; $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\vec{v}$ is an eigenvector of A. What is the corresponding eigenvalue?

Solution: use the defining property of an eigenvector:

$$A\,\vec{v} = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 2 & 2 & 2 & 2 \\ 5 & 1 & 0 & 2 \\ 4 & -2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+3-1+4 \\ 2+2+2+2 \\ 5+1+0+2 \\ 4-2+3+3 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 8\,\vec{v}.$$

Thus $\vec{v}$ is an eigenvector of the matrix A with corresponding eigenvalue $\lambda = 8$.

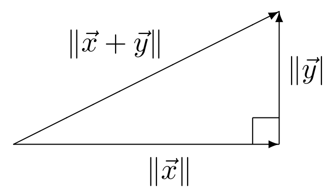
3. [avg: 6.41/10] Let $\vec{x}$ and $\vec{y}$ be two non-zero vectors in $\mathbb{R}^n$.

(a) [5 marks] Prove that if $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ then $\vec{x}$ and $\vec{y}$ are orthogonal. Illustrate this geometrically.

Solution:

$$\begin{aligned} & \|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 \\ \Rightarrow\ & (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} \\ \Rightarrow\ & \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} = \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} \\ \Rightarrow\ & \vec{x}\cdot\vec{y} = 0, \end{aligned}$$

so $\vec{x}$ and $\vec{y}$ are orthogonal.



$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$

The Pythagorean Theorem

(b) [5 marks] Prove that if $\vec{x}$ and $\vec{y}$ are orthogonal, then $\|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|$. Illustrate this geometrically.

Solution: if $\vec{x}\cdot\vec{y} = 0$, then

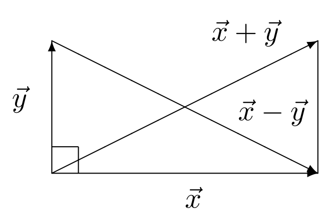
$$\|\vec{x}+\vec{y}\|^2 = \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

and

$$\|\vec{x}-\vec{y}\|^2 = \vec{x}\cdot\vec{x} - 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

Thus $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}-\vec{y}\|^2$; taking square roots, we get

$$\|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|.$$



The diagonals of a rectangle have the same length.

4. [avg: 4.58/10] Let A be an $m \times k$ matrix; let B be a $k \times n$ matrix.

(a) [3 marks] Show that $\text{nullity}(B) \leq \text{nullity}(A\,B)$. Recall: if X is a matrix, $\text{nullity}(X) = \dim(\text{null}(X))$.

Solution: show $\text{null}(B)$ is contained in $\text{null}(A\,B)$, and the result follows by taking dimensions.

$$\begin{aligned} \vec{x} \in \text{null}(B) \quad &\Rightarrow \quad B(\vec{x}) = \vec{0} \\ &\Rightarrow \quad A(B(\vec{x})) = A(\vec{0}) = \vec{0} \\ &\Rightarrow \quad x \in \text{null}(A\,B) \end{aligned}$$

Then $\dim(\text{null}(B)) \leq \dim(\text{null}(A\,B)$; or equivalently, $\text{nullity}(B) \leq \text{nullity}(A\,B)$.

(b) [3 marks] Use the Rank Theorem to show that $\text{rank}(AB) \leq \text{rank}(B)$.

Solution: use the result from part (a) and the Rank Theorem: $A\,B$ is an $m \times n$ matrix and so

$$\begin{aligned} \text{nullity}(B) \leq \text{nullity}(A\,B) \quad &\Rightarrow \quad n - \text{rank}(B) \leq n - \text{rank}(A\,B) \\ &\Rightarrow \quad \text{rank}(A\,B) \leq \text{rank}(B). \end{aligned}$$

(c) [4 marks] Show that if $\text{rank}(A) = k$ then $A^T A$ is invertible. (Hint: suppose $A^T A\,\vec{x} = \vec{0}$.)

Solution: follow the hint. $A^T A$ is $k \times k$; suppose $\vec{x}$ is in $\mathbb{R}^k$.

$$\begin{aligned} A^T A\,\vec{x} = \vec{0} \quad &\Rightarrow \quad \vec{x}^T A^T A\,\vec{x} = \vec{x}^T \vec{0} = 0 \\ &\Rightarrow \quad (A\,\vec{x})^T A\,\vec{x} = 0 \\ &\Rightarrow \quad (A\,\vec{x}) \cdot (A\,\vec{x}) = 0 \\ &\Rightarrow \quad \|A\,\vec{x}\|^2 = 0 \\ &\Rightarrow \quad A\,\vec{x} = \vec{0} \\ &\Rightarrow \quad \vec{x} = \vec{0}, \text{ since } \text{rank}(A) = k \Leftrightarrow \text{nullity}(A) = 0 \end{aligned}$$

Thus $\text{null}(A^T A) = \{\vec{0}\}$, and so $A^T A$ is invertible.

OR: Show that if A is an $m \times m$ invertible matrix, then $\vec{x}^T A^T A\,\vec{x} > 0$ for all $\vec{x} \neq \vec{0}$ in $\mathbb{R}^m$.

Solution: similar to above. $\vec{x}^T A^T A\,\vec{x} = (A\vec{x})^T (A\vec{x}) = (A\vec{x}) \cdot (A\vec{x}) = \|A\vec{x}\|^2 \geq 0$. But since A is invertible, $A\vec{x} = \vec{0}$ if and only if $\vec{x} = \vec{0}$. Thus for $\vec{x} \neq \vec{0}$, $\vec{x}^T A^T A\,\vec{x} > 0$.

5. [avg: 9.13/10] Solve the system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= 6x - y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= 2x + 3y \end{cases}$$

for x and y as functions of t, given that $(x, y) = (5, 7)$ when $t = 0$.

Solution: let $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$; find the eigenvalues and eigenvectors of A.

Eigenvalues: $\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 6 & 1 \\ -2 & \lambda - 3 \end{bmatrix} = \lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5)$. So the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 5$.

Eigenvectors:

$$\text{null} \begin{bmatrix} 4-6 & 1 \\ -2 & 4-3 \end{bmatrix} = \text{null} \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}; \text{ take } \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{null} \begin{bmatrix} 5-6 & 1 \\ -2 & 5-3 \end{bmatrix} = \text{null} \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} = \text{null} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}; \text{ take } \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution to the system of differential equations is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \vec{v}_1 e^{4t} + c_2 \vec{v}_2 e^{5t} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

To find c_1, c_2, let $t = 0$ and solve the system of linear equations

$$\begin{bmatrix} 5 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = - \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus the solutions to the system of differential equations is

$$x = 2e^{4t} + 3e^{5t} \text{ and } y = 4e^{4t} + 3e^{5t}.$$

6. [avg: 6.63/10] If $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

Step 1: find the eigenvalues of A.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{bmatrix} = \det \begin{bmatrix} \lambda - 2 & 3 - \lambda & 3 - \lambda \\ 1 & \lambda - 3 & 0 \\ 1 & 0 & \lambda - 3 \end{bmatrix}$$

$$= (\lambda - 3)^2 \det \begin{bmatrix} \lambda - 2 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = (\lambda - 3)^2(\lambda - 2 + 0 + 0 - 0 + 1 + 1) = \lambda(\lambda - 3)^2$$

Thus the eigenvalues of A are $\lambda_1 = 3$, repeated, and $\lambda_2 = 0$.

Step 2: find an *orthogonal* basis of eigenvectors for each eigenspace.

$$\text{null} \begin{bmatrix} 3-2 & 1 & 1 \\ 1 & 3-2 & 1 \\ 1 & 1 & 3-2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\};$$

$$\text{null} \begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} = \text{null} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & -3 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Step 3: for the columns of P, take the unit, orthogonal eigenvectors and for the diagonal entries of D take the corresponding eigenvalues:

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

7. [avg: 4.14/10] Let $S = \text{span}\left\{[\,1\ \ 1\ \ 0\ \ 1\,]^T, [\,0\ \ 1\ \ 1\ \ 1\,]^T, [\,-1\ \ 1\ \ 1\ \ 1\,]^T\right\}$.

(a) [3 marks] Find a basis for $S^\perp$, the orthogonal complement of S.

Solution:

$$S^\perp = \text{null}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Take $\vec{w} = [\,0\ \ -1\ \ 0\ \ 1\,]^T$ as the basis for $S^\perp$.

(b) [4 marks] Find the standard matrix of the linear transformation $\text{perp}_S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$.

Solution: let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ be the standard basis of $\mathbb{R}^4$. Then $\|\vec{w}\|^2 = 2$ and

$$\left.\begin{aligned} \text{perp}_S(\vec{e}_1) &= \text{proj}_{S^\perp}(\vec{e}_1) = \frac{\vec{e}_1 \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w} = \vec{0} \\ \text{perp}_S(\vec{e}_2) &= \text{proj}_{S^\perp}(\vec{e}_2) = \frac{\vec{e}_2 \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w} = -\frac{1}{2}\vec{w} \\ \text{perp}_S(\vec{e}_3) &= \text{proj}_{S^\perp}(\vec{e}_3) = \frac{\vec{e}_3 \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w} = \vec{0} \\ \text{perp}_S(\vec{e}_4) &= \text{proj}_{S^\perp}(\vec{e}_4) = \frac{\vec{e}_4 \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w} = \frac{1}{2}\vec{w} \end{aligned}\right\} \Rightarrow [\text{perp}_S] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Alternately, and more efficiently:

$$\text{perp}_S(\vec{x}) = \text{proj}_{S^\perp}(\vec{x}) = \frac{\vec{x} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w} = \frac{(-x_2 + x_4)}{2}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ x_2 - x_4 \\ 0 \\ -x_2 + x_4 \end{bmatrix},$$

from which you can read off the matrix.

(c) [3 marks] What is the standard matrix of $\text{proj}_S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$? (You can make use of part (b).)

Solution: use the fact that $[\text{proj}_S] = I - [\text{perp}_S]$. Thus

$$[\text{proj}_S] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

8. [avg: 7.71/10] Experimental data for the response, y, of an electronic device to an input, x in millivolts, is listed in the following table:

Trial i	1	2	3	4	5
Input x_i	2	3	4	5	6
Response y_i	5	7	8	11	12

Find the best fitting line with equation $y = a + b\,x$ for the given data.

Solution: let $M = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}$; $\vec{y} = \begin{bmatrix} 5 \\ 7 \\ 8 \\ 11 \\ 12 \end{bmatrix}$. The normal equations are $M^T M \begin{bmatrix} a \\ b \end{bmatrix} = M^T \vec{y}$. Solve:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 8 \\ 11 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 20 \\ 20 & 90 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 43 \\ 190 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 90 & -20 \\ -20 & 5 \end{bmatrix} \begin{bmatrix} 43 \\ 190 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 18 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 43 \\ 190 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 18 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 4.3 \\ 19 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 1.8 \end{bmatrix}$$

So the line that best fits the data has equation $y = 1.4 + 1.8\,x$.

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