University of Toronto
Solutions to **MAT188H1S TERM TEST**
of **Wednesday, March10, 2010**
Duration: 90 minutes

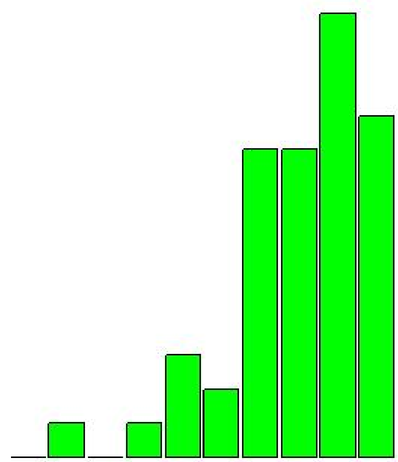
General Comments about the Test:

1. Regarding True or False questions: you can't prove a statement is true by giving an example; you must give a general argument. On the other hand, to prove a statement is false it is sufficient to give one counter example.

2. This test was designed to be a pretty straightforward test. And the results seem to bear that out.

Breakdown of Results: 48 students wrote this test. The marks ranged from 13.3% to 100%, and the average was 75.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	20.8%
A	47.9%	80-89%	27.1%
B	18.7%	70-79%	18.7%
C	18.7%	60-69%	18.7%
D	4.2%	50-59%	4.2%
F	10.5%	40-49%	6.3%
		30-39%	2.1%
		20-29%	0.0%
		10-19%	2.1%
		0-9%	0.0%



1. [8 marks] The parts of this question are unrelated.

   (a) [3 marks] Find the scalar equation of the plane passing through the point $P(3, 2, -1)$ and parallel to the plane with scalar equation $x + 2y + 3z = 0$.

   Solution: Take

$$\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   and use the point $P(3, 2, -1)$. Then the equation of the plane is

$$x + 2y + 3z = 3 + 4 - 3 = 4.$$

   (b) [5 marks] Find the volume of the parallelepiped in 3 dimensions determined by the three vectors

$$\vec{u} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   Solution: use $V = |\vec{u} \cdot \vec{v} \times \vec{w}| = \left|\det \left[\begin{array}{c|c|c} \vec{u} & \vec{v} & \vec{w} \end{array} \right]\right|$. Thus

$$\begin{aligned} V &= \left| \det \begin{bmatrix} 3 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \right| \\ &= \left| \det \begin{bmatrix} 3 & -3 & 4 \\ 0 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \right| \\ &= \left| \det \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \right| \\ &= |-7| \\ &= 7 \end{aligned}$$

2. [8 marks] Find the minimum distance between the skew lines $\mathbb{L}_1$ and $\mathbb{L}_2$ if their parametric equations are

$$\mathbb{L}_1 : \left\{ \begin{array}{ccccc} x & = & 4 & - & 2t \\ y & = & 2 & + & t \\ z & = & 1 & + & t \end{array} \right. \text{ and } \mathbb{L}_2 : \left\{ \begin{array}{ccccc} x & = & 2 & + & 3s \\ y & = & 2 & - & s \\ z & = & 3 & + & 2s \end{array} \right.$$

Solution: The line $\mathbb{L}_1$ passes through the point $P(4,2,1)$ with direction vector

$$\vec{d_1} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix};$$

the line $\mathbb{L}_2$ passes through the point $Q(2,2,3)$ with direction vector

$$\vec{d_2} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

A vector orthogonal to both lines is

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}.$$

Then the minimum distance between the two skew lines is given by

$$\|\text{proj}_{\vec{n}} \overrightarrow{PQ}\| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{\|\vec{n}\|} = \frac{|(3)(-2) + (7)(0) + (-1)(2)|}{\sqrt{9 + 49 + 1}} = \frac{|-8|}{\sqrt{59}} = \frac{8}{\sqrt{59}}.$$

3. [10 marks] Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$.

Solution:

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & -2 & 0 \\ -1 & \lambda - 1 & 2 \\ 0 & 1 & \lambda - 1 \end{bmatrix} \\ &= (\lambda - 1)^3 - 4(\lambda - 1) \\ &= (\lambda - 1)\left((\lambda - 1)^2 - 4\right) \\ &= (\lambda - 1)(\lambda^2 - 2\lambda - 3) \\ &= (\lambda - 1)(\lambda - 3)(\lambda + 1) \end{aligned}$$

So the **eigenvalues** of A are $\lambda_1 = 1, \lambda_2 = 3$ and $\lambda_3 = -1$.

Eigenvectors:

$$(I - A|O) = \left[\begin{array}{rrr|r} 0 & -2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; \text{ take } X_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$(3I - A|O) = \left[\begin{array}{rrr|r} 2 & -2 & 0 & 0 \\ -1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; \text{ take } X_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

$$(-I - A|O) = \left[\begin{array}{rrr|r} -2 & -2 & 0 & 0 \\ -1 & -2 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; \text{ take } X_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

4. [10 marks] Find both $f_1(x)$ and $f_2(x)$ if

$$\begin{cases} f_1' &= -f_1 &+& 5f_2 \\ f_2' &= f_1 &+& 3f_2 \end{cases} \quad \text{and } f_1(0) = 1;\, f_2(0) = -1.$$

Solution: let $A = \begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$. Then

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+1 & -5 \\ -1 & \lambda-3 \end{bmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2).$$

Eigenvalues of A : $(\lambda-4)(\lambda+2) = 0 \Leftrightarrow \lambda = 4$ or $\lambda = -2$.

Eigenvectors of A :

$$(4I - A|0) \to \left[\begin{array}{rr|r} 5 & -5 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{take } X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

and

$$(-2I - A|0) \to \left[\begin{array}{rr|r} -1 & -5 & 0 \\ -1 & -5 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{take } X_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-2x}.$$

To find c_1 and c_2, let $x = 0$:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Leftrightarrow c_1 = -\frac{2}{3},\ c_2 = -\frac{1}{3}.$$

Thus,

$$f_1(x) = -\frac{2}{3}e^{4x} + \frac{5}{3}e^{-2x}$$

and

$$f_2(x) = -\frac{2}{3}e^{4x} - \frac{1}{3}e^{-2x}.$$

5. Let $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a projection on the line with equation $y = 3x$; let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a projection on the line with equation $y = -\dfrac{1}{3}x$. Let the standard matrix of S be A; let the standard matrix of T be B.

(a) [9 marks] Find A and B, and calculate both AB and BA.

Solution: recall the formula from the book. A projection on the line $y = mx$ has matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$$

So with $m = 3$,

$$A = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

and with $m = -1/3$,

$$B = \frac{1}{1+1/9}\begin{bmatrix} 1 & -1/3 \\ -1/3 & 1/9 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}.$$

Then

$$AB = \frac{1}{100}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

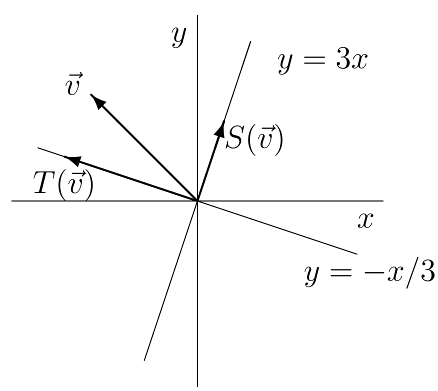
$$BA = \frac{1}{100}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) [2 marks] BONUS: Interpret the result from part (a) geometrically.

Solution: The lines $y = 3x$ and $y = -x/3$ are orthogonal to each other. For any vector $\vec{v}$, $S(\vec{v})$ is parallel to the line $y = 3x$, and $T(\vec{v})$ is parallel to the line $y = -x/3$.
Thus:
$S(\vec{v})$ is orthogonal to the line $y = -x/3$, and $T(v)$ is orthogonal to the line $y = 3x$.
Hence,

$$T \circ S(\vec{v}) = \vec{0} \text{ and } S \circ T(\vec{v}) = \vec{0}.$$

This means the matrix of $T \circ S$, namely BA, and the matrix of $S \circ T$, namely AB, are both the zero matrix.



6. [6 marks] The parts of this question are unrelated.

(a) [3 marks] Find the matrix that represents a reflection in the line $y = 3x$.

Solution: recall the formula from the book. A reflection in the line $y = mx$ has matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

So with $m = 3$, the matrix is

$$\frac{1}{10}\begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}.$$

(b) [3 marks] Find the matrix that represents a rotation of $\theta = 120°$ counterclockwise about the origin.

Solution: recall the formula from the book. A rotation of θ counterclockwise about the origin has matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

So with $\theta = 120°$, the matrix is

$$\begin{bmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}.$$

7. [9 marks; 3 marks for each part] Indicate if the following statements are True or False, and give a brief explanation why.

(a) If A is a 5×5 matrix such that $A^T = -A$, then $\det(A) = 0$. **True** **False**

Solution: True.

$$\begin{aligned} A^T = -A &\Rightarrow \det(A^T) = \det(-A) \\ &\Rightarrow \det A = (-1)^5 \det A \\ &\Rightarrow \det A = -\det A \\ &\Rightarrow \det A = 0 \end{aligned}$$

(b) If A is an $n \times n$ matrix that satisfies $A^3 - A = 2I$, then

$$A^{-1} = \frac{1}{2}(A^2 - I).$$

Solution: True.

$$\begin{aligned} A^3 - A = 2I &\Rightarrow A(A^2 - I) = 2I \\ &\Rightarrow A\left(\frac{1}{2}(A^2 - I)\right) = I \\ &\Rightarrow A^{-1} = \frac{1}{2}(A^2 - I) \end{aligned}$$

(c) If $\vec{d} \neq \vec{0}$ and $k \neq 0$, then $\text{proj}_{k\vec{d}}(\vec{v}) = \text{proj}_{\vec{d}}(\vec{v})$. **True** **False**

Solution: True.

$$\begin{aligned} \text{proj}_{k\vec{d}}(\vec{v}) &= \frac{\vec{v} \cdot k\vec{d}}{\|k\vec{d}\|^2} k\vec{d} \\ &= \frac{k^2}{|k^2|} \frac{\vec{v} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} \\ &= \frac{\vec{v} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} \\ &= \text{proj}_{\vec{d}}(\vec{v}) \end{aligned}$$