MAT188H1S - Linear Algebra - Winter 2015

Solutions to Term Test - March 4, 2015

Time allotted: 110 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

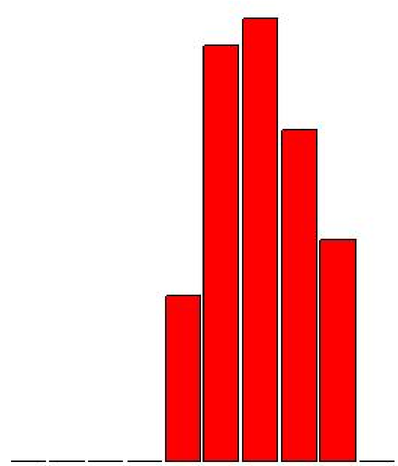
This test consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. Every question had a passing average except the subspace question, Question 5.
2. The subspace questions was done very badly. Some students are totally confused between sets, subspaces, matrices, vectors and numbers. These confusions are not peculiar to Question 5 though; they surfaced in many other questions. Such confusion costs you marks–as does bad notation.

Breakdown of Results: 57 students wrote this test. The marks ranged from 45% to 87.5%, and the average was 64.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0%
A	14.0%	80-89%	14.0%
B	21.1%	70-79%	21.1%
C	28.1%	60-69%	28.1%
D	26.3%	50-59%	26.3%
F	10.5%	40-49%	10.5%
		30-39%	0.0%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



PART I : No explanation is necessary.

1. [avg: 8.9/10] Given that the reduced echelon form of

$$A = \begin{bmatrix} 2 & 6 & -2 & 1 & 8 \\ 3 & 9 & -3 & 5 & 5 \\ 1 & 3 & -1 & 4 & -3 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 3 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. Put your answers in the blanks to the right.

(a) [1 mark] dim(row(A)) $\underline{\quad 2 \quad}$

(b) [1 mark] dim(col(A)) $\underline{\quad 2 \quad}$

(c) [1 mark] dim(null(A)) $\underline{\quad 3 \quad}$

(d) [2 marks] A basis for the row space of A

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

(e) [2 marks] A basis for the column space of A

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \right\}$$

(f) [3 marks] A basis for the null space of A

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

2. [avg: 8.8/10] Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ and $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be linear transformations defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \\ 2x_1 + 5x_2 \\ -x_1 + 4x_2 \end{bmatrix}, \quad S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 - x_2 + x_3 - x_4 \end{bmatrix}.$$

Put your answers to the following questions in the blanks to the right.

(a) [1 mark] What is the matrix of T? $\begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 2 & 5 \\ -1 & 4 \end{bmatrix}$

(b) [1 mark] What is the matrix of S? $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

(c) [1 mark] Is T one-to-one? Yes

(d) [1 mark] Is T onto? No

(e) [1 mark] Is S one-to-one? No

(f) [1 mark] Is S onto? Yes

(g) [2 marks] Is $S \circ T$ one-to-one, where $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$? Yes

Because the matrix of $S \circ T$ is $\begin{bmatrix} 4 & 11 \\ 2 & 5 \end{bmatrix}$, which is invertible.

(h) [2 marks] Is $T \circ S$ onto, where $(T \circ S)(\mathbf{x}) = T(S(\mathbf{x}))$? No

Because the matrix of $T \circ S$ is $\begin{bmatrix} 4 & -2 & 4 & -2 \\ 1 & 3 & 1 & 3 \\ 7 & -3 & 7 & 3 \\ 3 & -5 & 3 & -5 \end{bmatrix}$, which has dependent columns.

Continued...

PART II : Present **COMPLETE** solutions to the following questions in the space provided.

3. [avg: 8.6/10] Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y \\ x + 4y \end{bmatrix}.$$

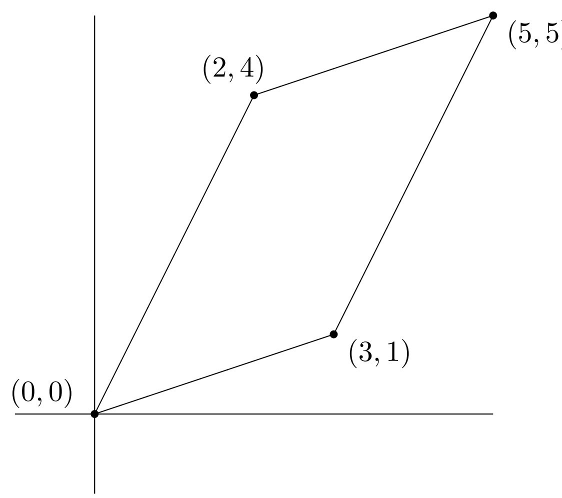
(a) [5 marks] Draw the image of the unit square[1] under T and label all four vertices.

Solution: the image of the unit square is the parallelogram determined by

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$



(b) [5 marks] Find $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

Solution: if the matrix of T is A then the matrix of T^{-1} is A^{-1}. We have

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}; \quad A^{-1} = \frac{1}{12-2}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}.$$

So

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4x - 2y \\ -x + 3y \end{bmatrix}.$$

[1] The unit square is the square with the four vertices $(0,0), (1,0), (0,1), (1,1)$.

4. [avg: 7.6/10] Let

$$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -6 \\ 3 \\ -3 \end{bmatrix}.$$

Find A^{-1} and use it to solve the equation $A\,\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

Solution: use the Gaussian algorithm, but to avoid fractions start with $(3A|3I)$:

$$(3A|3I) = \left[\begin{array}{rrr|rrr} 2 & 2 & -1 & 3 & 0 & 0 \\ -1 & 2 & 2 & 0 & 3 & 0 \\ 2 & -1 & 2 & 0 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} -1 & 2 & 2 & 0 & 3 & 0 \\ 2 & 2 & -1 & 3 & 0 & 0 \\ 2 & -1 & 2 & 0 & 0 & 3 \end{array}\right] \sim$$

$$\left[\begin{array}{rrr|rrr} -1 & 2 & 2 & 0 & 3 & 0 \\ 0 & 6 & 3 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 & 6 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} -1 & 2 & 2 & 0 & 3 & 0 \\ 0 & 2 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} -1 & 2 & 2 & 0 & 3 & 0 \\ 0 & 2 & 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & -1 & 2 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{rrr|rrr} 3 & -6 & 0 & -2 & -5 & 4 \\ 0 & 6 & 0 & 4 & 4 & -2 \\ 0 & 0 & 3 & -1 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 3 & 0 & 0 & 2 & -1 & 2 \\ 0 & 3 & 0 & 2 & 2 & -1 \\ 0 & 0 & 3 & -1 & 2 & 2 \end{array}\right] = (3I|3A^{-1});$$

So

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}, \text{ which just happens to be } A^T.$$

Then

$$\mathbf{x} = A^{-1}\,\mathbf{b} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} -6 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ 2 \end{bmatrix}.$$

5. [avg: 1.1/10] Let A be an invertible $n \times n$ matrix; let $S = \{\mathbf{x} \in \mathbb{R}^n \mid A^T\,\mathbf{x} = A^{-1}\,\mathbf{x}\}$.

(a) [6 marks] Show that S is a subspace of $\mathbb{R}^n$. (It is *not* necessary to use the definition of subspace.)

Solution: the easy way to do this is to observe that the null space of any matrix is always a subspace, and that

$$S = \text{null}(A^T - A^{-1}).$$

(b) [4 marks] Let $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find the dimension of S.

Solution:

$$\begin{aligned}
\text{rank}(A^T - A^{-1}) &= \text{rank}\left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T - \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}\right) \\
&= \text{rank}\left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \\
&= \text{rank}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 24/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
&= \text{rank}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = 2;
\end{aligned}$$

so $\dim(S) = \text{nullity}(A^T - A^{-1}) = 4 - \text{rank}(A^T - A^{-1}) = 4 - 2 = 2$.

6. [avg: 5.2/10] Suppose A is a 17×12 matrix with rank equal to 8.

   (a) [4 marks] Let T be the linear transformation defined by $T(\mathbf{x}) = A\,\mathbf{x}$, for all $\mathbf{x}$ in $\mathbb{R}^{12}$. What is the dimension of the kernel of T?

   Solution: $\dim(\ker(T)) = \text{nullity}(A) = 12 - \text{rank}(A) = 12 - 8 = 4$.

   (b) [6 marks; 2 for each part] Let S be the linear transformation defined by $S(\mathbf{y}) = A^T\,\mathbf{y}$, for all $\mathbf{y}$ in $\mathbb{R}^{17}$. Determine if the following statements could be true.

   Note: A^T is a 12×17 matrix, so $S : \mathbb{R}^{17} \longrightarrow \mathbb{R}^{12}$.

   (*i*) S is onto.

   Solution: cannot be true, since $\dim(\text{range}(S)) = \text{rank}(A^T) = \text{rank}(A) = 8 < 12$.

   (*ii*) S is one-to-one.

   Solution: cannot be true,
   since $\dim(\ker(S)) = \text{nullity}(A^T) = 17 - \text{rank}(A^T) = 17 - 8 = 9 > 0$.

   (*iii*) $S \circ T$ is onto, with T as in part (a) above.

   Solution: cannot be true. The matrix of $S \circ T$ is $A^T A$, which is a 12×12 matrix. By the Big Theorem, $S \circ T$ is onto if and only if it is one-to-one. But $S \circ T$ cannot be one-to-one, because T isn't, by part (a). That is, there is a non-zero vector $\mathbf{x} \in \mathbb{R}^{12}$ such that $T(\mathbf{x}) = \mathbf{0}$. Then $\mathbf{x} \neq \mathbf{0}$, and

   $$(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = S(\mathbf{0}) = \mathbf{0},$$

   which means $\ker(S \circ T) \neq \{\mathbf{0}\}$.

7. [avg; 6.3/10] Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation such that

$$T(\mathbf{v}_1) = 3\mathbf{v}_1 - 4\mathbf{v}_2 \text{ and } T(\mathbf{v}_2) = \mathbf{v}_1 + 2\mathbf{v}_2.$$

Find the matrix of T.

Solution: let the matrix of T be A. Then

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = A\,\mathbf{v}_1 = T(\mathbf{v}_1) = 3\mathbf{v}_1 - 4\mathbf{v}_2 = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

and

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = A\,\mathbf{v}_2 = T(\mathbf{v}_2) = \mathbf{v}_1 + 2\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Combining these two equations into one matrix equation gives

$$\begin{aligned} A \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix} \quad &\Leftrightarrow \quad A = \begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\ &\Leftrightarrow \quad A = \begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &\Leftrightarrow \quad A = \frac{1}{2} \begin{bmatrix} 8 & 6 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}. \end{aligned}$$

8. [avg: 5.2/10] Indicate if the following statements are **True** or **False**, and give a *brief* explanation why.

   (a) [2 marks] If A and B are 2×2 matrices then $(A+B)(A-B) = A^2 - B^2$. **False**

   Solution: its only True if $AB = BA$, so not true in general.

   (b) [2 marks] If A is a 2×2 matrix then A and A^2 have the same rank. **False**

   Solution: let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$; then $A^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   So A has rank 1 but A^2 has rank 0.

   (c) [2 marks] $\dim\left(\operatorname{span}\left\{\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}\right\}\right) = 3$ **False**

   Solution: the dimension is only 2 since the three given vectors are linearly dependent:

$$\begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

   (d) [2 marks] $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 3 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -2 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \end{bmatrix}\right\} = \mathbb{R}^4$ **False**

   Solution: need at least 4 independent vectors to span $\mathbb{R}^4$, but given set contains at most 3 independent vectors since:

$$\begin{bmatrix} 4 \\ 6 \\ -2 \\ 10 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \\ -1 \\ 5 \end{bmatrix}; \quad \begin{bmatrix} 4 \\ 9 \\ 0 \\ 11 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \\ -1 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

   (e) [2 marks] If A and B are invertible $n \times n$ matrices then AB is also invertible. **True**

   Solution: $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$; that is, $(AB)^{-1} = B^{-1}A^{-1}$.

This page is for rough work; it will not be marked, unless you have given specific indication on some previous page to look at this page.