**MAT188H1S Linear Algebra, March 16, 2016**
Time Alloted: 110 minutes
Aids permitted: Casio FX-991 or Sharp EL-520 calculator
**Solutions to test:**

**1.** Given that the reduced echelon form of

$$A = \begin{bmatrix} 2 & -4 & 1 & 4 & 1 \\ 3 & -6 & 3 & 9 & 0 \\ 1 & -2 & 4 & 9 & -3 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following.

(a) [1 mark] $\dim(\text{row}(A))$ Answer: $\dim(\text{row}(A)) = \text{rank}(R) = 2$

(b) [1 mark] $\dim(\text{col}(A))$ Answer: $\dim(\text{col}(A)) = \text{rank}(R) = 2$

(c) [1 mark] $\dim(\text{null}(A))$ Answer: $\dim(\text{null}(A)) = 5 - \text{rank}(R) = 3$

(d) [2 marks] A basis for the row space of $A$

Answer: the non-zero rows of $R$,

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \right\},$$

or any two independent rows of $A$.

(e) [2 marks] A basis for the column space of $A$

Answer: the two columns of $A$ corresponding to the pivot columns of $R$,

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\},$$

or any two independent columns of $A$.

(f) [3 marks] A basis for the null space of $A$

Solution: $A\mathbf{x} = \mathbf{0} \Leftrightarrow R\mathbf{x} = \mathbf{0}$. Let $x_2 = s, x_4 = t, x_5 = u$ be parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - t - u \\ s \\ -2t + u \\ t \\ u \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for null$(A)$ is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

**2.** Decide if the following statements are **True** or **False**, and give a *brief* explanation why.

(a) [2 marks] If $A$ is a $5 \times 5$ matrix then $\text{rank}(A) \neq \text{nullity}(A)$.

**True:**

$\text{rank}(A) = \text{nullity}(A) = r \Rightarrow 2r = 5$. This is impossible, since $r$ cannot be a fraction.

(b) [2 marks] If $A$ is an $n \times n$ invertible matrix then $A^T$ is also invertible.

**True:**

since $\det(A) = \det(A^T)$, it follows that $\det(A) \neq 0 \Leftrightarrow \det(A^T) \neq 0$.

(c) [2 marks] $\dim \left( \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\} \right) = 3$

**True:**

the given vectors are independent since $\det \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = 3 \neq 0.$

(d) [2 marks] $\dim \left( \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -2 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \end{bmatrix} \right\} \right) = 5$

**False:**

since any subspace $S$ of $\mathbf{R}^4$ must have dimension less than or equal to 4.

(e) [2 marks] $\det \begin{bmatrix} 1 & 0 & -2 & 4 \\ 4 & 2 & -8 & 3 \\ -1 & 3 & 2 & 7 \\ 3 & 1 & -6 & 9 \end{bmatrix} = 0.$

**True:**

since the third column of the given matrix is $-2$ times the first column of the given matrix.

**3.** Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4x + 2y \\ 7x + 3y \end{bmatrix}.$$

(a) [5 marks] Draw the image of the unit square[1] under $T$ and label all four vertices. What is its area?

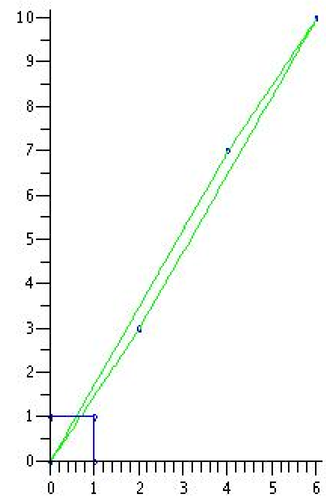
**Solution:** the matrix of $T$ is

$$A = \begin{bmatrix} 4 & 2 \\ 7 & 3 \end{bmatrix}.$$

The vertices of the image of the unit square are

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 10 \end{bmatrix}.$$

See diagram to the right. (Sorry: its a long, skinny parallelogram!) The area of this parallelogram is $|\det(A)| = |12 - 14| = 2$.




(b) [5 marks] Find $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

**Solution:** the matrix of $T^{-1}$ is

$$A^{-1} = \begin{bmatrix} 4 & 2 \\ 7 & 3 \end{bmatrix}^{-1} = \frac{1}{12 - 14}\begin{bmatrix} 3 & -2 \\ -7 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 & 2 \\ 7 & -4 \end{bmatrix}.$$

So

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -3 & 2 \\ 7 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3x + 2y \\ 7x - 4y \end{bmatrix}.$$

[1] The unit square is the square with the four vertices $(0,0), (1,0), (0,1), (1,1)$.

**4.** Use any method you like to solve the system of equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ -3x_1 + 7x_2 - 6x_3 = -2 \\ 2x_1 - 3x_2 = -1 \end{cases}.$$

**Method 1:** good old row reduction on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ -3 & 7 & -6 & -2 \\ 2 & -3 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right];$$

so the solution is $x_1 = 1, x_2 = 1, x_3 = 1$.

**Method 2:** find the inverse of the coefficient matrix, $A$. Using the Gaussian algorithm:

$$[A \mid I] = \left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 6 & 1 & -1 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -18 & -3 & 5 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}\right] = \left[I \mid A^{-1}\right]$$

Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

**Method 3:** use Cramer's Rule. First observe that $\det(A) = 0 + 24 + 9 - 14 - 18 - 0 = 1$. Thus:

$$x_1 = \det \begin{bmatrix} 0 & -2 & 1 \\ -2 & 7 & -6 \\ -1 & -3 & 0 \end{bmatrix} = 0 - 12 + 6 + 7 - 0 - 0 = 1;$$

$$x_2 = \det \begin{bmatrix} 1 & 0 & 1 \\ -3 & -2 & -6 \\ 2 & -1 & 0 \end{bmatrix} = 0 + 0 + 3 + 4 - 6 - 0 = 1;$$

$$x_3 = \det \begin{bmatrix} 1 & -2 & 0 \\ -3 & 7 & -2 \\ 2 & -3 & -1 \end{bmatrix} = -7 + 8 + 0 - 0 - 6 + 6 = 1.$$

**5.** Find all values of the parameter $a$ such that the system of equations

$$\begin{cases} 4x_1 & & + & ax_3 & = & 8 \\ -x_1 & + & ax_2 & + & 3x_3 & = & 1 \\ ax_1 & & + & 9x_3 & = & 9 \end{cases}$$

has $(i)$ no solution; $(ii)$ a unique solution; $(iii)$ infinitely many solutions.

**Solution:** let the coefficient matrix of the system of equations be

$$A = \begin{bmatrix} 4 & 0 & a \\ -1 & a & 3 \\ a & 0 & 9 \end{bmatrix}.$$

Then $\det(A) = a(36 - a^2) = a(6 - a)(6 + a)$.

$(ii)$ : the system will have a unique solution if $A$ is invertible: if $a \neq 0, a \neq 6, a \neq -6$.

That leaves three cases left, which must be checked separately.

$(iii)$ : if $a = 0$ the system becomes

$$\begin{cases} 4x_1 & & & = & 8 \\ -x_1 & & + & 3x_3 & = & 1 \\ & & & 9x_3 & = & 9 \end{cases},$$

which has infinitely many solutions: $x_1 = 2, x_2 = t, x_3 = 1$, where $t$ is a parameter.

$(i)$ : there turn out to be no solutions to the system if $a = \pm 6$, evidenced by a row in the reduced augmented matrix that looks like [0 0 0 | non-zero].

$a = 6$:

$$\left[\begin{array}{ccc|c} 4 & 0 & 6 & 8 \\ -1 & 6 & 3 & 1 \\ 6 & 0 & 9 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 6 & 3 & 1 \\ 4 & 0 & 6 & 8 \\ 6 & 0 & 9 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 6 & 3 & 1 \\ 0 & 24 & 18 & 12 \\ 0 & 36 & 27 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 6 & 3 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right].$$

$a = -6$:

$$\left[\begin{array}{ccc|c} 4 & 0 & -6 & 8 \\ -1 & -6 & 3 & 1 \\ -6 & 0 & 9 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & -6 & 3 & 1 \\ 4 & 0 & -6 & 8 \\ -6 & 0 & 9 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & -6 & 3 & 1 \\ 0 & -24 & 6 & 12 \\ 0 & 36 & -9 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 6 & -3 & -1 \\ 0 & 4 & -1 & -2 \\ 0 & 0 & 0 & 7 \end{array}\right].$$

**6.** Suppose $A$ is a $7 \times 4$ matrix with rank equal to 3.

(a) [4 marks] Let $T$ be the linear transformation defined by $T(\mathbf{x}) = A\,\mathbf{x}$, for all $\mathbf{x}$ in $\mathbf{R}^4$. What is the dimension of the kernel of $T$?

**Solution:** use the Rank-Nullity Theorem.

$$\dim(\ker(T)) = \text{nullity}(A) = 4 - \text{rank}(A) = 4 - 3 = 1.$$

(b) [6 marks; 2 for each part] Let $S$ be the linear transformation defined by $S(\mathbf{y}) = A^T\,\mathbf{y}$, for all $\mathbf{y}$ in $\mathbf{R}^7$. Decide if the following three statements are true or false.

**Solution:** note that $S : \mathbf{R}^7 \longrightarrow \mathbf{R}^4$ and its matrix is $4 \times 7$.

$(i)$ $S$ is onto.

**False:** $\dim(\text{range}(S)) = \text{rank}(A^T) = \text{rank}(A) = 3 < 4.$

$(ii)$ $S$ is one-to-one.

**False:** $\dim(\text{kernel}(S)) = \text{nullity}(A^T) = 7 - \text{rank}(A^T) = 7 - 3 = 4 > 0.$

$(iii)$ $S \circ T$ is onto, with $T$ as in part (a) above.

**False:** by part (a) there is a non-zero vector $\mathbf{v}$ in $\mathbf{R}^4$ such that

$$T(\mathbf{v}) = \mathbf{0}.$$

Thus also

$$(S \circ T)(\mathbf{v}) = S(T(\mathbf{v})) = S(\mathbf{0}) = \mathbf{0},$$

which means $S \circ T$ is not one-to-one. Since

$$S \circ T : \mathbf{R}^4 \longrightarrow \mathbf{R}^4,$$

it is also not onto, by the Big Theorem.

**7.** Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ 2x_1 - x_2 + 3x_3 \\ 5x_1 + x_2 + 11x_3 \\ x_1 + x_2 + 3x_3 \end{bmatrix}.$$

Find a basis for *(i)* the kernel of $T$, and *(ii)* the range of $T$.

**Solution:** If the matrix of $T$ is $A$, then $\ker(T) = \text{null}(A)$ and $\text{range}(T) = \text{col}(A)$. We have

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 5 & 1 & 11 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 6 & 6 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

So a basis for $\text{range}(T)$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Actually, any two columns of $A$ will do. As for the kernel of $T$ : we need to solve the homogeneous system $A\,\mathbf{x} = \mathbf{0}$. Let $x_3 = t$ be a parameter. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix};$$

hence a basis for $\ker(T)$ is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

**8.** Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ X = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \ Y = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

(a) [5 marks] Verify by direct calculation—show your work!—that

$$X^2 = 3X, \ Y^2 = -X \text{ and } XY = 3Y = YX.$$

**Solution:**

$$X^2 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4+1+1 & -2-2+1 & -2+1-2 \\ -2-2+1 & 1+4+1 & 1-2-2 \\ -2+1-2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} = 3X.$$

$$Y^2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1-1 & 1 & 1 \\ 1 & -1-1 & 1 \\ 1 & 1 & -1-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = -X.$$

$$XY = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+1 & -2-1 \\ -2-1 & -1+1 & 1+2 \\ 1+2 & -1-2 & 1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & -3 \\ -3 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix} = 3Y.$$

Similarly for $YX$.

(b) [5 marks] Let $A = I + a\,X + b\,Y$, for some scalars $a$ and $b$. Find conditions on $a$ and $b$ such that

$$A^{-1} = A^T.$$

**Solution:**

$$\begin{aligned} AA^T = I \ &\Rightarrow \ (I + aX + bY)(I + aX + b\ Y)^T = I \\ &\Rightarrow \ (I + aX + bY)(I^T + aX^T + bY^T) = I \\ &\Rightarrow \ (I + aX + bY)(I + aX - bY) = I \\ &\Rightarrow \ I + aX + bY + aX + a^2X^2 + baYX - bY - abXY - b^2Y^2 = I \\ &\Rightarrow \ I + 2aX + 3a^2X + b^2X = I \\ &\Rightarrow \ (2a + 3a^2 + b^2)X = O \\ &\Rightarrow \ 2a + 3a^2 + b^2 = 0 \end{aligned}$$