MAT188H1S - Linear Algebra

Solutions to Term Test - Monday, March 13, 2017

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

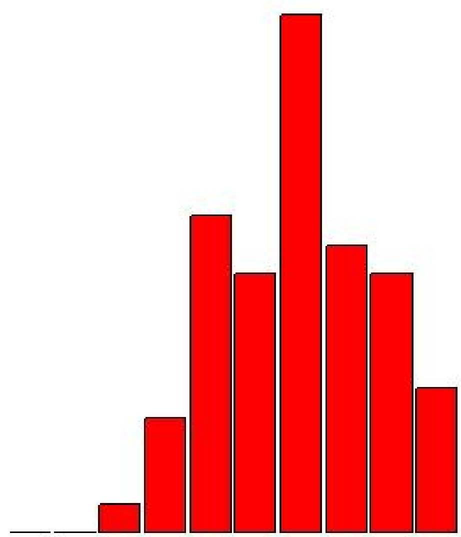
General Comments:

1. Every question has a passing average, but Question 1 should have had a much higher average!

2.

Breakdown of Results: 68 students wrote this test. The marks ranged from 22.7% to 98.7%, and the average was 64.2%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	7.5%
A	20.9%	80-89%	13.4%
B	14.9%	70-79%	14.9%
C	26.9%	60-69%	26.9%
D	13.4%	50-59%	13.4%
F	23.9%	40-49%	16.4%
		30-39%	6.0%
		20-29%	1.5%
		10-19%	0.0%
		0-9%	0.0%



1. [avg: 9.54/15] Given that the reduced row echelon form of

$$A = \begin{bmatrix} 3 & 6 & 1 & 5 & 5 \\ 4 & 8 & 1 & 6 & 7 \\ -1 & -2 & 1 & 1 & -3 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. (No justification is required.)

(a) [1 mark] the rank of A Answer: 2

(b) [1 mark] $\dim(\text{Row}(A))$ Answer: 2

(c) [1 mark] $\dim(\text{Col}(A))$ Answer: 2

(d) [1 mark] $\dim(\text{Null}(A))$ Answer: 3

(e) [1 mark] $\dim(\text{Row}(A^T))$ Answer: 2

(f) [1 mark] $\dim(\text{Col}(A^T))$ Answer: 2

(g) [1 mark] $\dim(\text{Null}(A^T))$ Answer: 1

(h) [2 marks] A basis for the row space of A.

Solution: any two independent rows of A or R will do: Answer: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$

(i) [2 marks] A basis for the column space of A.

Solution: any two independent columns of A will do: Answer: $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(j) [4 marks] A basis for the null space of A.

Solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - t - 2u \\ s \\ -2t + u \\ t \\ u \end{bmatrix}$ Answer: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

2. [avg: 6.99/10] Consider the plane in $\mathbb{R}^3$ that passes through the three points $P(1,1,0)$, $Q(2,-1,1)$ and $R(-1,3,-4)$.

(a) [7 marks] Find the scalar equation of the plane.

Solution: for the normal vector take $\overrightarrow{PQ} \times \overrightarrow{PR}$ (or $\overrightarrow{QR} \times \overrightarrow{QP}$, or $\overrightarrow{RP} \times \overrightarrow{RQ}$.)

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}.$$

Then use any of the three given points P, Q or R, to calculate the point-normal form of the equation:

$$6x_1 + 2x_2 - 2x_3 = 6 \cdot 1 + 2 \cdot 1 - 2 \cdot 0 = 8 \Leftrightarrow 3x_1 + x_2 - x_3 = 4.$$

(b) [3 marks] What is the area of the triangle with vertices P, Q, R?

Solution: use your calculations from part (a).

$$\text{Area of } \Delta PQR = \frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\left\|\begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}\right\| = \frac{\sqrt{44}}{2} = \sqrt{11}.$$

3. [avg: 7.68/10] Write the system of equations

$$\begin{cases} x_1 + x_2 = -7 \\ 2x_1 + 4x_2 + x_3 = -16 \\ x_1 + 2x_2 + x_3 = 9 \end{cases}$$

in matrix form, $A\vec{x} = \vec{b}$. Then solve it by finding and using the inverse of the coefficient matrix A.

Solution:

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -16 \\ 9 \end{bmatrix}; \text{ or simply state } A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -7 \\ -16 \\ 9 \end{bmatrix}.$$

Use the Gaussian algorithm to find A^{-1} :

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right] = [I|A^{-1}]$$

Then $\vec{x} = A^{-1}\vec{b}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -7 \\ -16 \\ 9 \end{bmatrix} = \begin{bmatrix} 11 \\ -18 \\ 34 \end{bmatrix}$$

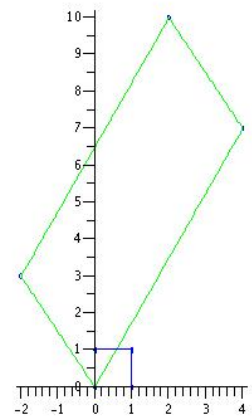
4. [avg: 7.06/10] Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 2x_2 \\ 7x_1 + 3x_2 \end{bmatrix}.$$

(a) [4 marks] Draw the image of the unit square[1] under L and label all four vertices.

Solution:

$$[L] = \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 7 & 3 \end{bmatrix}$$



(b) [6 marks] Find $L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

Solution: use the formula for the inverse of a 2×2 matrix.

$$[L^{-1}] = [L]^{-1} = \begin{bmatrix} 4 & -2 \\ 7 & 3 \end{bmatrix}^{-1} = \frac{1}{12+14}\begin{bmatrix} 3 & 2 \\ -7 & 4 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 3 & 2 \\ -7 & 4 \end{bmatrix}.$$

Thus

$$L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{26}\begin{bmatrix} 3 & 2 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 3x_1 + 2x_2 \\ -7x_1 + 4x_2 \end{bmatrix}.$$

[1] The unit square is the square with the four vertices $(0,0), (1,0), (0,1), (1,1)$.

5. [avg: 5.0/10] Consider the linear transformations $S_1, S_2, S_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined geometrically as:

S_1 is a rotation of $\pi/4$ counterclockwise around the origin;

S_2 is projection onto the vector $\vec{v} = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$;

S_3 is a reflection in the line passing through the origin with direction vector $\vec{d} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$.

(a) [2 marks] Write down the matrix of S_1.

Solution: the rotation matrix with $\theta = \pi/4$:

$$[S_1] = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(b) [3 marks] Write down the matrix of S_2.

Solution: use the projection formula.

$$[S_2] = \begin{bmatrix} \operatorname{proj}_{\vec{d}}(\vec{e}_1) & \operatorname{proj}_{\vec{d}}(\vec{e}_1) \end{bmatrix} = \begin{bmatrix} \frac{3}{13}\begin{pmatrix} 3 \\ 2 \end{pmatrix} & \frac{2}{13}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix}.$$

(c) [3 marks] Write down the matrix of S_3.

Solution: a reflection matrix with $m = -1$ (or with $\theta = 3\pi/2$):

$$[S_3] = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

$$[S_3] = \begin{bmatrix} \cos(3\pi/2) & \sin(3\pi/2) \\ \sin(3\pi/2) & -\cos(3\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

OR: use $S_3(x) = \operatorname{refl}_{\vec{n}}(\vec{x}) = \vec{x} - 2\operatorname{proj}_{\vec{n}}(\vec{x})$ where $\vec{n} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so that

$$S_3(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{2(x_1+x_2)}{1^2+1^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(d) [2 marks] What is the matrix of the linear transformation obtained by first applying S_2, then applying S_3, and then applying S_1?

Solution: use $[S_1 \circ S_3 \circ S_2] = [S_1][S_3][S_2]$; so the matrix of the composition is

$$[S_1][S_3][S_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \frac{1}{13} \begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix} = \frac{1}{13\sqrt{2}} \begin{bmatrix} 3 & 2 \\ -15 & -10 \end{bmatrix}$$

6. [avg: 5.62/10] Suppose A is a 8×5 matrix with rank equal to 3.

(a) [4 marks] Let L be the linear transformation defined by $L(\vec{x}) = A\,\vec{x}$, for all $\vec{x}$ in $\mathbb{R}^5$ What is the dimension of the nullspace (or kernel) of L?

Solution: $\dim(\text{Null}(L)) = \dim(\text{Null}(A)) = 5 - \text{rank}(A) = 5 - 3 = 2.$

(b) [6 marks; 3 for each part] Let K be the linear transformation defined by $K(\vec{y}) = A^T\,\vec{y}$, for all $\vec{y}$ in $\mathbb{R}^8$. Decide if the following two statements are true or false.

(i) the range of K is $\mathbb{R}^5$

Solution: False. $K : \mathbb{R}^8 \longrightarrow \mathbb{R}^5$. The range of K is $\text{Col}(A^T)$. So

$$\dim(\text{range}(K)) = \dim(\text{Col}(A^T)) = \text{rank}(A^T) = \text{rank}(A) = 3 < 5,$$

and the range of K cannot be $\mathbb{R}^5$.

(ii) the nullspace of K is $\{\vec{0}\}$.

Solution: False. Use $\text{Null}(K) = \text{Null}(A^T)$. We have

$$\dim(\text{Null}(A^T)) = 8 - \text{rank}(A^T) = 8 - 3 = 5 > 0;$$

so $\text{Null}(K) \neq \{\vec{0}\}$.

7. [avg: 6.26/10] Consider the plane Π in $\mathbb{R}^3$ with scalar equation $2x_1 - 3x_2 - 5x_3 = 7$ and the point with coordinates $Q(2, -3, -1)$. Find both

(a) [5 marks] the minimum distance from the point Q to the plane.

(b) [5 marks] the point on the plane Π closest to the point Q.

You can solve either part (a) or part (b) first; its up to you.

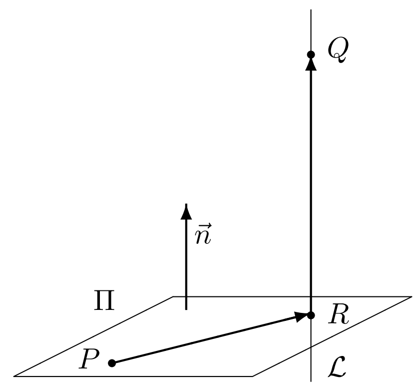
Solution: setting things up.

- The normal vector to Π is $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}$.
- Let R be the point on Π that is closest to Q.
- Suppose P is any point on the plane Π.
- Then

$$\overrightarrow{RQ} = \text{proj}_{\vec{n}} \left(\overrightarrow{PQ}\right)$$

and

$$\overrightarrow{PR} = \text{perp}_{\vec{n}} \left(\overrightarrow{PQ}\right).$$



For calculations, pick the point P to be $(1, 0, -1)$. (But any point on Π would do.)

(a) Then the minimum distance from Q to the plane is given by $D = \|\overrightarrow{RQ}\| = \|\text{proj}_{\vec{n}} \left(\overrightarrow{PQ}\right)\|$. We have

$$\overrightarrow{PQ} = \begin{bmatrix} 2-1 \\ -3-0 \\ -1-(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

and

$$D = \|\text{proj}_{\vec{n}} \left(\overrightarrow{PQ}\right)\| = \left\| \frac{2+9+0}{4+9+25} \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix} \right\| = \frac{11}{38}\sqrt{38} = \frac{11}{\sqrt{38}}.$$

(b) Let R have coordinates (r_1, r_2, r_3). Then $\overrightarrow{QR} = -\text{proj}_{\vec{n}} \left(\overrightarrow{PQ}\right)$, so that

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} - \frac{11}{38} \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix} = \frac{1}{38} \begin{bmatrix} 54 \\ -81 \\ 17 \end{bmatrix}.$$

Alternate Calculations:
Then $\overrightarrow{PR} = \text{perp}_{\vec{n}}\left(\overrightarrow{PQ}\right) = \overrightarrow{PQ} - \text{proj}_{\vec{n}}\left(\overrightarrow{PQ}\right)$. Rearranging gives $\overrightarrow{PR} + \overrightarrow{QP} = -\text{proj}_{\vec{n}}\left(\overrightarrow{PQ}\right)$, whence $\overrightarrow{QR} = -\text{proj}_{\vec{n}}\left(\overrightarrow{PQ}\right)$, so that

$$\Leftrightarrow \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} - \frac{11}{38}\begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix} = \frac{1}{38}\begin{bmatrix} 54 \\ -81 \\ 17 \end{bmatrix}.$$

Alternate Solution: let $\mathcal{L}$ be the line that passes through the point Q and is perpendicular to the plane Π. Then $\mathcal{L}$ has parametric equations

$$\begin{cases} x_1 &= 2+2t \\ x_2 &= -3-3t \\ x_3 &= -1-5t \end{cases}$$

The closest point on Π to the point Q is the point R, which is the intersection of the line $\mathcal{L}$ and the plane Π. To find this intersection point substitute from the line into the plane, and solve for t:

$$2(2+2t) - 3(-3-3t) - 5(-1-5t) = 7 \Leftrightarrow t = -\frac{11}{38}.$$

Then:
(b) R has coordinates

$$\left(2 - 2\left(\frac{11}{38}\right), -3 + 3\left(\frac{11}{38}\right), -1 + 5\left(\frac{11}{38}\right)\right) = \left(\frac{54}{38}, -\frac{81}{38}, \frac{17}{38}\right)$$

and (a), the minimum distance from the point Q to the plane Π is

$$\|\overrightarrow{QR}\| = \left\| -\frac{11}{38}\begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} \right\| = \frac{11}{38}\sqrt{4+9+25} = \frac{11}{\sqrt{38}}$$

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

The end.