MAT188H1S - Linear Algebra

Solutions to Term Test - Wednesday, March 14, 2018

Time allotted: 110 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

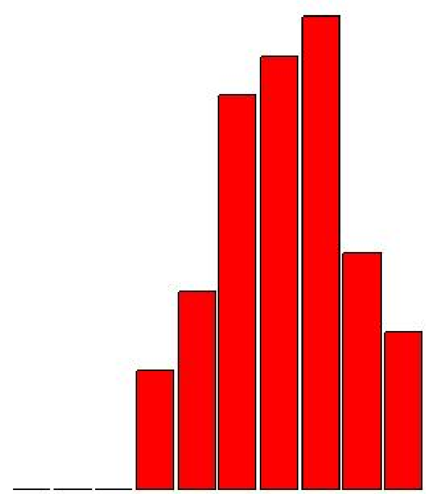
General Comments::

1. The computational questions, #1 through #6, were well done. But the two theoretical questions, #7 and #8, were poorly done.

2. There were many students who exhibited bad notation: confusing $\Rightarrow$ with $=$, or misusing $=$ by putting equal signs between things that *aren't* equal, i.e. a linear transformation and a matrix, or a vector and a number.

Breakdown of Results: 51 students wrote this test. The marks ranged from 30% to 97.5%, and the average was 65.3%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	7.8%
A	19.6%	80-89%	11.8%
B	23.5%	70-79%	23.5%
C	21.6%	60-69%	21.6%
D	19.6%	50-59%	19.6%
F	15.7%	40-49%	9.8%
		30-39%	5.9%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



1. [avg: 8.86/10] Given that the reduced row echelon form of

$$A = \begin{bmatrix} 3 & 9 & 1 & 8 & 8 \\ 4 & 12 & 2 & 10 & 12 \\ -1 & -3 & 1 & -4 & 0 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 3 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. (No justification is required.)

(a) [1 mark] the rank of A Answer: 2

(b) [1 mark] $\dim(\text{Row}(A))$ Answer: 2

(c) [1 mark] $\dim(\text{Col}(A))$ Answer: 2

(d) [1 mark] $\dim(\text{Null}(A))$ Answer: 3

(e) [2 marks] A basis for the row space of A.

Answer: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

(f) [2 marks] A basis for the column space of A.

Answer: $\left\{ \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

(g) [2 marks] A basis for the null space of A.

Solution: let $x_2 = s, x_4 = t, x_5 = u$ be parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3s - 3t - 2u \\ s \\ t - 2u \\ t \\ u \end{bmatrix}.$$

Answer: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

2. [avg: 8.35/10] Consider the plane in $\mathbb{R}^3$ that passes through the three points

$$P(1,1,-1),\ Q(2,1,1) \text{ and } R(0,3,-3).$$

(a) [7 marks] Find the scalar equation of the plane.

Solution: take the normal of the plane to be $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$. We have

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ \overrightarrow{PR} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix},\ \vec{n} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}.$$

Pick P as a point on the plane; its scalar equation is

$$-4x_1 + 2x_3 = -4(1) + 2(-1) = -6 \Leftrightarrow 2x_1 - x_3 = 3.$$

(b) [3 marks] What is the area of the triangle with vertices P, Q, R?

Solution: the area of a triangle is half the area of the parallelogram formed by two sides of the triangle. Thus

$$\text{area of } \Delta PQR = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\left\| \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\| = \sqrt{5}.$$

3. [avg: 7.41/10] Write the system of equations

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & -8 \\ -x_1 & + & 3x_2 & + & x_3 & = & 16 \\ x_1 & + & 2x_2 & - & x_3 & = & 24 \end{array}\right.$$

in matrix form, $A\vec{x} = \vec{b}$. Then solve it by finding and using the inverse of the coefficient matrix A.

Solution: in matrix form the system of equations can be written as

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & 2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -8 \\ 16 \\ 24 \end{bmatrix}}_{\vec{b}}.$$

Use the Gaussian algorithm to find A^{-1} :

$$[A|I] = \left[\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 2 & 4 & 0 & 1 & 0 & 1 \\ 0 & 5 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 10 & 0 & 0 & 5 & -4 & 1 \\ 0 & 5 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 10 & 0 & 0 & 5 & -4 & 1 \\ 0 & 10 & 0 & 0 & 2 & 2 \\ 0 & 0 & 10 & 5 & 0 & -5 \end{array}\right] = [10\,I|10\,A^{-1}].$$

Thus

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 5 & -4 & 1 \\ 0 & 2 & 2 \\ 5 & 0 & -5 \end{bmatrix},$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\vec{b} = \frac{1}{10}\begin{bmatrix} 5 & -4 & 1 \\ 0 & 2 & 2 \\ 5 & 0 & -5 \end{bmatrix}\begin{bmatrix} -8 \\ 16 \\ 24 \end{bmatrix} = \begin{bmatrix} -8 \\ 8 \\ -16 \end{bmatrix}.$$

4. [avg: 7.59/10] Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

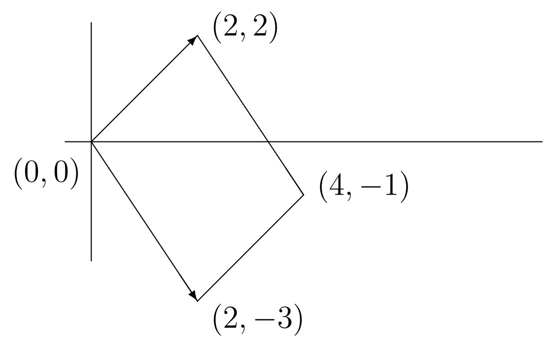
$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_2 \\ -3x_1 + 2x_2 \end{bmatrix}.$$

(a) [5 marks] Draw the image of the unit square[1] under L and label all four vertices.

Solution: $L(\vec{0}) = \vec{0}$ and

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}; L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix};$$

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$



(b) [5 marks] Find $L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

Solution: use $[L^{-1}] = [L]^{-1}$. The matrix of L is

$$\begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix},$$

so

$$[L^{-1}] = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}^{-1} = \frac{1}{10}\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix},$$

by the formula for the inverse of a 2×2 matrix, as proved in class. Then

$$L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2x_1 - 2x_2 \\ 3x_1 + 2x_2 \end{bmatrix}.$$

[1] The unit square is the square with the four vertices $(0,0), (1,0), (0,1), (1,1)$.

5. [avg: 6.75/10] Find the matrices of the following linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$:

(a) [2 marks] a rotation of $\pi/6$ counterclockwise around the origin.

Solution: use the formula for the rotation matrix. The matrix is

$$\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

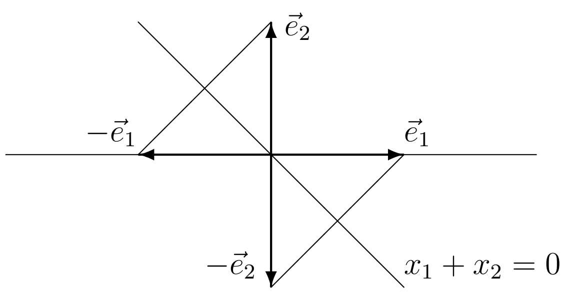
(b) [3 marks] a projection onto the vector $\vec{v} = \begin{bmatrix} -3 & 2 \end{bmatrix}^T$;

Solution: calculate the matrix by projecting $\vec{e}_1$ and $\vec{e}_2$ onto $\vec{v}$:

$$\left.\begin{aligned} \text{proj}_{\vec{v}}(\vec{e}_1) &= \frac{\vec{e}_1 \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = -\frac{3}{13}\begin{bmatrix} -3 \\ 2 \end{bmatrix} \\ \text{proj}_{\vec{v}}(\vec{e}_2) &= \frac{\vec{e}_2 \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \frac{2}{13}\begin{bmatrix} -3 \\ 2 \end{bmatrix} \end{aligned}\right\} \Rightarrow [\text{proj}_{\vec{v}}] = \frac{1}{13}\begin{bmatrix} 9 & -6 \\ -6 & 4 \end{bmatrix}$$

(c) [3 marks] a reflection in the line with equation $x_1 + x_2 = 0$.

Solution: find the images of $\vec{e}_1$ and $\vec{e}_2$ geometrically:



So the matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(d) [2 marks] the transformation of part (c) followed by the transformation of part (a).

Solution: multiply the two matrices together, in the correct order:

$$\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}.$$

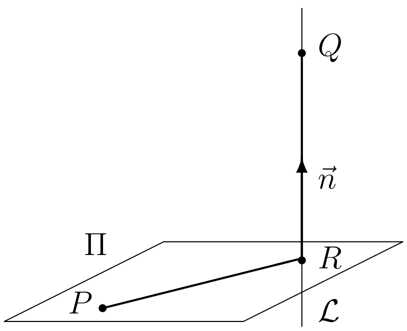
6. [avg: 8.04/10] Consider the plane Π in $\mathbb{R}^3$ with scalar equation $2x_1 + 6x_2 + 3x_3 = 11$ and the point with coordinates $Q(2, -3, -1)$. Find both

(a) [5 marks] the minimum distance from the point Q to the plane.

(b) [5 marks] the point on the plane Π closest to the point Q.

You can solve either part (a) or part (b) first; its up to you.

What we know: see the figure below and the observations to the right.



- Let $\vec{n}$ be the normal vector to the plane Π.
- Let P be any point on Π.
- Then the minimum distance from Q to the plane is given by $D = \|\text{proj}_{\vec{n}}\left(\overrightarrow{PQ}\right)\|$.
- Let R be the point on the plane Π that is closest to Q.
- Let $\mathcal{L}$ be the line orthogonal to the plane Π and passing through the point Q.
- R is the point of intersection of $\mathcal{L}$ and Π.

Solution: we have $\vec{n} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$. Take $(1, 1, 1)$ as a point P on the plane; then $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}$.

(a) the minimum distance from the point Q to the plane Π is given by

$$D = \|\text{proj}_{\vec{n}}\left(\overrightarrow{PQ}\right)\| = \left\| \frac{(2 - 24 - 6)}{49}\,\vec{n} \right\| = \frac{4}{7}\|\vec{n}\| = \frac{4}{7}\sqrt{4 + 36 + 9} = 4.$$

(b) The vector equation of $\mathcal{L}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}.$$

Substitute from the line $\mathcal{L}$ into the equation of the plane Π to find the value of the parameter t at the intersection point, R:

$$2(2 + 2t) + 6(-3 + 6t) + 3(-1 + 3t) = 11 \Leftrightarrow 49\,t = 28 \Leftrightarrow t = \frac{4}{7}.$$

Thus the coordinates of R, the point on the plane closest to Q, are

$$(x_1, x_2, x_3) = \left(2 + \frac{8}{7}, -3 + \frac{24}{7}, -1 + \frac{12}{7}\right) = \left(\frac{22}{7}, \frac{3}{7}, \frac{5}{7}\right).$$

7. [avg: 2.2/10] Suppose A is an $n \times n$ matrix such that $A^2 = O_{n,n}$. (That is, the square of A is the zero matrix.)

   (a) [5 marks] Prove that the column space of A is contained in the null space of A.

   Solution 1: I did this example in class, and here is the solution I gave.
   Let $A = \left[\begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{array} \right]$; that is, the i th column of A is $\vec{a}_i$. Then

   $$A^2 = A\,A = A \left[\begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{array} \right] = \left[\begin{array}{cccc} A\vec{a}_1 & A\vec{a}_2 & \ldots & A\vec{a}_n \end{array} \right],$$

   and so for each i, $1 \leq i \leq n$,

   $$A^2 = O \Rightarrow A\,\vec{a}_i = \vec{0} \Rightarrow \vec{a}_i \in \text{Null}(A).$$

   Since $\text{Null}(A)$ is a subspace of $\mathbb{R}^n$ it is closed under vector addition and scalar multiplication; hence

   $$\text{Col}(A) = \text{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \subset \text{Null}(A).$$

   Solution 2: here's an even nicer solution that one student came up with on the test.

   $$\begin{aligned} \vec{y} \in \text{Col}(A) \quad &\Rightarrow \quad \vec{y} = A\,\vec{x}, \text{ for some vector } \vec{x} \in \mathbb{R}^n \\ &\Rightarrow \quad A\,\vec{y} = A(A\,\vec{x}) = A^2\,\vec{x} = \vec{0}, \text{ since it is given that } A^2 = O \\ &\Rightarrow \quad \vec{y} \in \text{Null}(A) \\ &\Rightarrow \quad \text{Col}(A) \subset \text{Null}(A) \end{aligned}$$

   (b) [5 marks] Explain why $\text{rank}(A) \leq n/2$.

   Solution: use part (a) and the rank-nullity theorem.

   $$\begin{aligned} \text{Col}(A) \subset \text{Null}(A) \quad &\Rightarrow \quad \dim(\text{Col}(A)) \leq \dim(\text{Null}(A)) \\ &\Rightarrow \quad \text{rank}(A) \leq n - \text{rank}(A) \\ &\Rightarrow \quad 2\,\text{rank}(A) \leq n \\ &\Rightarrow \quad \text{rank}(A) \leq \frac{n}{2} \end{aligned}$$

8. [avg: 3.08/10] Let A be an $n \times n$ matrix and let $S = \{\vec{x} \in \mathbb{R}^n \mid A^2\,\vec{x} = A\,\vec{x}\}$.

(a) [5 marks] Show that S is a subspace of $\mathbb{R}^n$.

Solution 1: the easy way is to observe that $S = \mathrm{Null}(A^2 - A)$, and the null space of a matrix is a subspace, as proved in class.

Solution 2: you can use the definition of subspace.

1. S is non-empty: $A^2\,\vec{0} = \vec{0} = A\,\vec{0}$, so $\vec{0} \in S$.
2. S is close under vector addition: if $\vec{x}$ and $\vec{y}$ are both in S, then

$$A^2(\vec{x}+\vec{y}) = \underbrace{A^2\vec{x} + A^2\vec{y} = A\vec{x} + A\vec{y}}_{\text{since } \vec{x} \text{ and } \vec{y} \text{ are in } S} = A(\vec{x}+\vec{y}).$$

   Consequently, $\vec{x}+\vec{y}$ is in S.
3. S is closed under scalar multiplication: if $\vec{x}$ is in S and k is a scalar, then

$$A^2(k\vec{x}) = k\,\underbrace{A^2\vec{x} = kA\vec{x}}_{\text{since } \vec{x} \text{ in } S} = A(k\vec{x}).$$

   Consequently, $k\vec{x}$ is in S.

(b) [5 marks] Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$.

With S as defined in part (a), find a basis for S and state its dimension.

Solution:

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 6 \\ 12 & 12 & 12 \\ 18 & 18 & 18 \end{bmatrix};$$

$$A^2 - A = \begin{bmatrix} 6 & 6 & 6 \\ 12 & 12 & 12 \\ 18 & 18 & 18 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 5 \\ 10 & 10 & 10 \\ 15 & 15 & 15 \end{bmatrix}.$$

$$S = \mathrm{Null}(A^2 - A) = \mathrm{Null}\begin{bmatrix} 5 & 5 & 5 \\ 10 & 10 & 10 \\ 15 & 15 & 15 \end{bmatrix} = \mathrm{Null}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}.$$

Since these two spanning vectors are linearly independent—they are not parallel—they form a basis for S, and $\dim(S) = 2$.

OR you could observe that S is a plane passing through the origin, so its dimension is two.

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