University of Toronto
Solutions to **MAT 188H1F TERM TEST**
of **Thursday, November 15, 2007**
Duration: 60 minutes

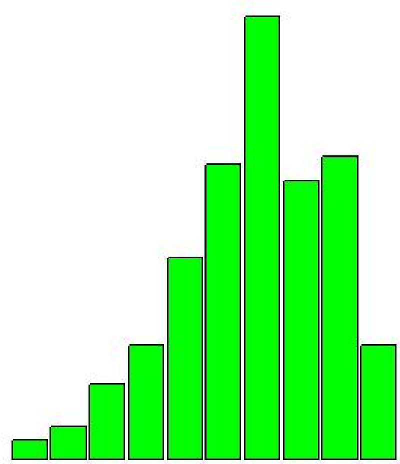
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

1. Questions 1, 2, 3 and 4 test some basic formulas and methods. These all show up in the homework problems.
2. Questions 5 and 6 are a quick survey of the some of the standard results about linear transformations on $\mathbb{R}^2$.
3. Question 7 consists of True or False questions about subspaces and spanning sets, as developed in Section 4.1
4. Some alternate solutions are included.

Breakdown of Results: 974 students wrote this test. The marks ranged from 0% to 100%, and the average was 62.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	6.1%
A	22.2%	80-89%	16.1%
B	14.9%	70-79%	14.9%
C	23.6%	60-69%	23.6%
D	15.7%	50-59%	15.7%
F	23.6%	40-49%	10.8%
		30-39%	6.1%
		20-29%	4.0%
		10-19%	1.7%
		0-9%	1.0 %



1. [6 marks] Find the area of the triangle with vertices

$$A(1,1,-1), B(3,3,2), C(5,0,-4).$$

Solution: Let $\vec{u} = \overrightarrow{AB} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \overrightarrow{AC} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$. Then the area of the triangle is given by

$$\begin{aligned} \frac{1}{2}\|\vec{u} \times \vec{v}\| &= \frac{1}{2}\left\| \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \right\| \\ &= \frac{1}{2}\left\| \begin{bmatrix} -3 \\ 18 \\ -10 \end{bmatrix} \right\| \\ &= \frac{1}{2}\sqrt{(-3)^2 + 18^2 + (-10)^2} \\ &= \frac{1}{2}\sqrt{433} \end{aligned}$$

Alternates: You could just as well use

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{bmatrix} 3 \\ -18 \\ 10 \end{bmatrix}$$

or

$$\overrightarrow{CB} \times \overrightarrow{CA} = \begin{bmatrix} 3 \\ -18 \\ 10 \end{bmatrix}$$

Alternate Solution: Use $A = \frac{1}{2} b \cdot h$. Take $b = \|\overrightarrow{AB}\|$. Use

$$h = \|\overrightarrow{AC}\| \sin\theta \text{ with } \cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} = -\frac{3}{\sqrt{442}}.$$

Then

$$A = \frac{1}{2}\|\overrightarrow{AB}\|\|\overrightarrow{AC}\| \sin\theta = \frac{1}{2}\sqrt{17}\sqrt{26}\sqrt{\frac{433}{442}} = \frac{\sqrt{433}}{2}.$$

2. [6 marks] Find the point on the plane with equation

$$x + 2y - 2z = -1$$

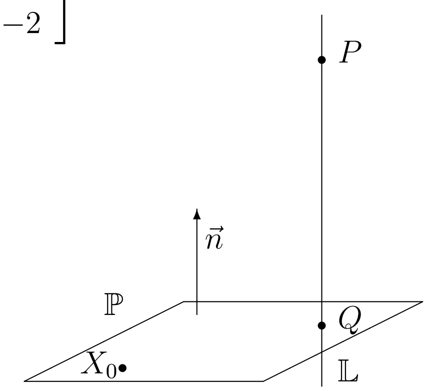
that is closest to the point $P(2, 2, -1)$.

Solution: Let the plane be $\mathbb{P}$; its normal is

$$\vec{n} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

The line $\mathbb{L}$ normal to the plane $\mathbb{P}$ passing through P has parametric equations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$



Let Q be the point on $\mathbb{P}$ closest to P; Q is the intersection of the line $\mathbb{L}$ and the plane $\mathbb{P}$. Substitute from the line into the plane:

$$\begin{aligned} (2+t) + 2(2+2t) - 2(-1-2t) = -1 \quad &\Leftrightarrow \quad 2 + t + 4 + 4t + 2 + 4t = -1 \\ &\Leftrightarrow \quad 9t = -9 \\ &\Leftrightarrow \quad t = -1 \end{aligned}$$

So the point Q has coordinates $(x, y, z) = (2 - 1, 2 - 2, -1 + 2) = (1, 0, 1)$.

Alternate Solution: $X_0(-1, 0, 0)$ is a point on $\mathbb{P}$. Then

$$\begin{aligned} \overrightarrow{X_0Q} + \operatorname{proj}_{\vec{n}}\left(\overrightarrow{X_0P}\right) = \overrightarrow{X_0P} \quad &\Leftrightarrow \quad \overrightarrow{X_0Q} = \overrightarrow{X_0P} - \operatorname{proj}_{\vec{n}}\left(\overrightarrow{X_0P}\right) \\ &\Leftrightarrow \quad \begin{bmatrix} x+1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - \frac{\overrightarrow{X_0P} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \\ &\Leftrightarrow \quad \begin{bmatrix} x+1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \\ &\Leftrightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-1 \\ 0-0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

So the coordinates of Q are $(x, y, z) = (1, 0, 1)$.

3. [6 marks] Let

$$L_1: \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -4 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, t \text{ a parameter}$$

and

$$L_2: \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 15 \end{bmatrix} + s \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, s \text{ a parameter}$$

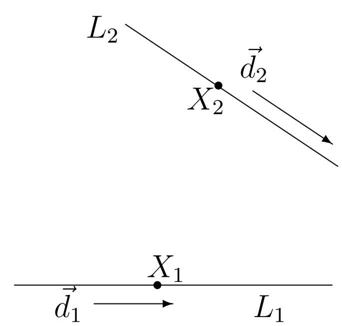
be two lines in $\mathbb{R}^3$. Find the minimum distance between L_1 and L_2.

Solution:

L_1 is parallel to $\vec{d}_1$ and passes through the point X_1; L_2 is parallel to $\vec{d}_2$ and passes through the point X_2. The minimum distance between the two lines is

$$D = \|\text{proj}_{\vec{n}} \left(\overrightarrow{X_1X_2}\right)\|,$$

where $\vec{n} = \vec{d}_1 \times \vec{d}_2$, which is normal to both L_1 and L_2.



We have

$$\overrightarrow{X_1X_2} = \begin{bmatrix} 12-3 \\ 0-(-8) \\ 15-(-4) \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 19 \end{bmatrix} \text{ and } \vec{n} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -12 \end{bmatrix}.$$

Then

$$\begin{aligned} D &= \|\text{proj}_{\vec{n}} \left(\overrightarrow{X_1X_2}\right)\| \\ &= \frac{|\overrightarrow{X_1X_2} \cdot \vec{n}|}{\|\vec{n}\|} \\ &= \frac{|27+32-228|}{\sqrt{169}} \\ &= 13 \end{aligned}$$

Alternate Solution: Find the two closest points. Solve

$$\begin{bmatrix} 9+4s \\ 8-3t \\ 19+s-t \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} 9+4s \\ 8-3t \\ 19+s-t \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = 0 \Leftrightarrow (s,t) = (-3,4).$$

So the two closest points are $P(0,0,12)$ and $Q(3,4,0)$ and $D = \|\overrightarrow{PQ}\| = 13$.

4. [10 marks] Find all the solutions, $f_1(x), f_2(x)$, if

$$\begin{cases} f_1' &= 2f_1 &+ &4f_2 \\ f_2' &= 3f_1 &+ &3f_2 \end{cases}$$

Solution: Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$; then

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & -4 \\ -3 & \lambda - 3 \end{bmatrix} = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1).$$

So the eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = -1$.

Find eigenvectors. (Note: I am using the notation from Section 4.1)

$$E_6(A) = \text{null}\,(6I - A) = \text{null} \begin{bmatrix} 4 & -4 \\ -3 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

and

$$E_{-1}(A) = \text{null}\,(-I - A) = \text{null} \begin{bmatrix} -3 & -4 \\ -3 & -4 \end{bmatrix} = \text{null} \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}.$$

Thus

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6x} + c_2 \begin{bmatrix} -4 \\ 3 \end{bmatrix} e^{-x};$$

that is,

$$f_1(x) = c_1 e^{6x} - 4c_2 e^{-x}$$

and

$$f_2(x) = c_1 e^{6x} + 3c_2 e^{-x}$$

for arbitrary constants c_1, c_2.

5. [7 marks] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x + 2y \end{bmatrix}.$$

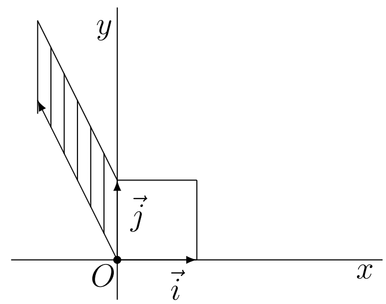
Plot the image of the unit square, and find the formula for $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

Solution:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

So the image of the unit square is the parallelogram determined by

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$



The standard matrix of T is

$$A = [T(\vec{i})|T(\vec{j})] = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.$$

Then the standard matrix of T^{-1} is

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix},$$

and the formula for T^{-1} is

$$\begin{aligned} T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 2x + y \\ -x \end{bmatrix} \end{aligned}$$

6. [7 marks] Find the standard matrix of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is: a reflection in the line $y = x$, followed by a rotation through $\pi/3$ counterclockwise, followed by another reflection in the line $y = x$. Is T a reflection or a rotation?

Solution: Note: I am using

$$Q_m = \frac{1}{2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \text{ and } R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where Q_m is the standard matrix of a reflection in the line $y = mx$ and R_θ is the standard matrix of a rotation through θ, counterclockwise.

Let the standard matrix of T be A. Then

$$\begin{aligned}
A &= Q_1\, R_{\pi/3}\, Q_1 \\
&= \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}\frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}.
\end{aligned}$$

T is a rotation (of $\pi/3$ clockwise); that is, $A = R_{-\pi/3}$. (It can't be a reflection since $\det A = 1$.)

7. [8 marks; 2 marks for each part] Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}.$$

Indicate if the following statements are True or False, and give a brief explanation why. Note: as in the textbook, 0 represents both the zero vector or the number zero, depending on context.

(a) $\operatorname{null} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

True: $(A|0) \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$;

so the basic solution of $AX = 0$ is $X = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(b) $\operatorname{im} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

True:

$$\begin{aligned} \operatorname{im} A &= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \\ &= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \text{ since } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

(c) $\{X \in \mathbb{R}^3 | AX = A^T X\}$ is a subspace of $\mathbb{R}^3$.

True: $\{X \in \mathbb{R}^3 | AX = A^T X\} = \operatorname{null}(A - A^T)$

(d) $\{X \in \mathbb{R}^3 | AX = 2X\} = \operatorname{span}\{0\}$

False: 2 is an eigenvalue of A, since

$$\det(2I - A) = \det \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -2 & -1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix} = 1(2-2) = 0.$$

So $E_2(A) \neq \operatorname{span}\{0\}$