University of Toronto
Solutions to **MAT188H1F TERM TEST**
of **Tuesday, November 9, 2010**
Duration: 100 minutes

General Comments about the Test:

1. Lots of students are still using incorrect notation

$$\Rightarrow \text{ instead of } \rightarrow, = \text{ instead of } \rightarrow, \text{ etc}$$

   or even worse, are equating a matrix to a determinant, or a transformation to a matrix, or a determinant to a vector, etc. If you keep doing it, we'll keep taking marks off.

2. Rational and irrational numbers are not the same things. What your calculator gives you is by default a rational number. So (from Question 3)

$$\frac{8}{\sqrt{59}} \neq 1.04$$

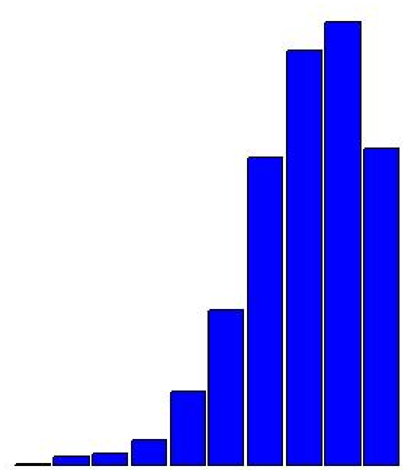
   One is an *approximation* of the other, so don't put equal signs between them. Respect equality!

3. For Question 7: use properties of determinants to ease the calculation of $\det(\lambda I - A)$. If you simply expand the determinant and get a cubic, then you have to factor it–and that might not be easy.

4. Many students need to explain their work; every time a marker has to figure out what you are doing it costs you marks.

Breakdown of Results: 950 students wrote this test. The marks ranged from 7% to 100%, and the average was 74.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	18.0%
A	43.2%	80-89%	25.2%
B	23.6 %	70-79%	23.6%
C	17.5%	60-69%	17.5 %
D	8.8%	50-59%	8.8 %
F	6.9%	40-49%	4.2%
		30-39%	1.5%
		20-29%	0.6%
		10-19%	0.5%
		0-9%	0.1%



1. [8 marks] Find the reduced row-echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\ 3 & 6 & 5 & -2 & 13 \\ -2 & -4 & -2 & 0 & -8 \\ -1 & -2 & 1 & -2 & 2 \end{bmatrix}.$$

What is the rank of A?

Solution:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\ 3 & 6 & 5 & -2 & 13 \\ -2 & -4 & -2 & 0 & -8 \\ -1 & -2 & 1 & -2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 2 & -2 & 4 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & -2 & 5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of A is the number of leading 1's in the reduced row-echelon matrix; so the rank is 3.

Note: The reduced row-echelon form of a matrix is unique; there is only one correct answer.

2. [9 marks] Consider the three points in space: $A(1,1,1), B(2,3,-5), C(0,2,2)$.

   (a) [5 marks] Find the scalar equation of the plane passing through the three points A, B and C.

   Solution: Let $\vec{u} = \overrightarrow{AB} = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$ and $\vec{v} = \overrightarrow{AC} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

   Then a normal vector to the plane is given by

$$\begin{aligned} \vec{n} = \vec{u} \times \vec{v} &= \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}; \end{aligned}$$

   and using the point C, the scalar equation of the plane is

$$\vec{n} \cdot \begin{bmatrix} x-0 \\ y-2 \\ z-2 \end{bmatrix} = 0 \Leftrightarrow 8x + 5y + 3z = 16.$$

   Alternates: You could just as well take

$$\vec{n} = \overrightarrow{BA} \times \overrightarrow{BC} = \begin{bmatrix} -8 \\ -5 \\ -3 \end{bmatrix}$$

   or

$$\vec{n} = \overrightarrow{CB} \times \overrightarrow{CA} = \begin{bmatrix} -8 \\ -5 \\ -3 \end{bmatrix}$$

   and you could use either of the given points, A, B or C.

   (b) [4 marks] What is the area of the triangle with vertices A, B and C?

   Solution: area $= \frac{1}{2}\|\vec{u} \times \vec{v}\| = \frac{1}{2}\sqrt{8^2 + 5^2 + 3^2} = \frac{\sqrt{98}}{2} = \frac{7}{\sqrt{2}}$.

3. [9 marks] Find the minimum distance between the skew lines $\mathbb{L}_1$ and $\mathbb{L}_2$ if their parametric equations are

$$\mathbb{L}_1 : \left\{ \begin{array}{ccccc} x & = & 4 & - & 2t \\ y & = & 2 & + & t \\ z & = & 1 & + & t \end{array} \right. \text{ and } \mathbb{L}_2 : \left\{ \begin{array}{ccccc} x & = & 2 & + & 3s \\ y & = & 2 & - & s \\ z & = & 3 & + & 2s \end{array} \right.$$

where s and t are parameters.

Solution: The line $\mathbb{L}_1$ passes through the point $P(4, 2, 1)$ with direction vector

$$\vec{d_1} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix};$$

the line $\mathbb{L}_2$ passes through the point $Q(2, 2, 3)$ with direction vector

$$\vec{d_2} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

A vector orthogonal to both lines is

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}.$$

Then the minimum distance between the two skew lines is given by

$$\|\text{proj}_{\vec{n}}\overrightarrow{PQ}\| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{\|\vec{n}\|} = \frac{|(3)(-2) + (7)(0) + (-1)(2)|}{\sqrt{9 + 49 + 1}} = \frac{|-8|}{\sqrt{59}} = \frac{8}{\sqrt{59}}.$$

Alternate Solution: this is longer, but you could actually find the two points that are closest together. Let

$$\vec{z} = \begin{bmatrix} -2 + 3s + 2t \\ -s - t \\ 2 + 2s - t \end{bmatrix}$$

be the general vector from a point on $\mathbb{L}_1$ to a point on $\mathbb{L}_2$. Then solve $\vec{z} \cdot \vec{d_1} = 0$ and $\vec{z} \cdot \vec{d_2} = 0$, or equivalently $\vec{z} \times \vec{n} = \vec{0}$, to find $s = -18/59, t = 74/59$. Then the minimum distance between the two lines is

$$\|\vec{z}\| = \left\| \begin{bmatrix} -24/59 \\ -56/59 \\ 8/59 \end{bmatrix} \right\| = \frac{\sqrt{3776}}{59} = \frac{8}{\sqrt{59}}.$$

4. [9 marks] Compute the following:

(a) [5 marks] $\det\begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 4 & 0 & 4 \\ -1 & 2 & 1 & 3 \\ 5 & 3 & 0 & 1 \end{bmatrix}$

Solution: use properties of determinants to simplify your work.

$$\det\begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 4 & 0 & 4 \\ -1 & 2 & 1 & 3 \\ 5 & 3 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 4 & 0 & 4 \\ 0 & 2 & 0 & 4 \\ 5 & 3 & 0 & 1 \end{bmatrix} = (-1)(-1)^4 \det\begin{bmatrix} 2 & 4 & 4 \\ 0 & 2 & 4 \\ 5 & 3 & 1 \end{bmatrix}$$

$$= (-1)(2)(2)\det\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 5 & 3 & 1 \end{bmatrix} = -4(1 + 20 + 0 - 10 - 0 - 6) = -20.$$

(b) [4 marks] $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 7 & 2 \\ 2 & -1 & 3 \end{bmatrix}^{-1}$

Solution: this solution uses the Gaussian algorithm.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 7 & 2 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 7 & 2 & 0 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 7 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -14 & 3 & 7 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -27 & 6 & 14 \\ 0 & 7 & 0 & -28 & 7 & 14 \\ 0 & 0 & 1 & 14 & -3 & -7 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -23 & 5 & 12 \\ 0 & 1 & 0 & -4 & 1 & 2 \\ 0 & 0 & 1 & 14 & -3 & -7 \end{array}\right]$$

So

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 7 & 2 \\ 2 & -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -23 & 5 & 12 \\ -4 & 1 & 2 \\ 14 & -3 & -7 \end{bmatrix}.$$

Alternate Solution: use the adjoint formula:

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{[C_{ij}]^T}{(-1)} = (-1)\begin{bmatrix} 23 & 4 & -14 \\ -5 & -1 & 3 \\ -12 & -2 & 7 \end{bmatrix}^T = \begin{bmatrix} -23 & 5 & 12 \\ -4 & 1 & 2 \\ 14 & -3 & -7 \end{bmatrix}.$$

5. [8 marks] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by the formula

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+4y \\ x+3y \end{bmatrix}.$$

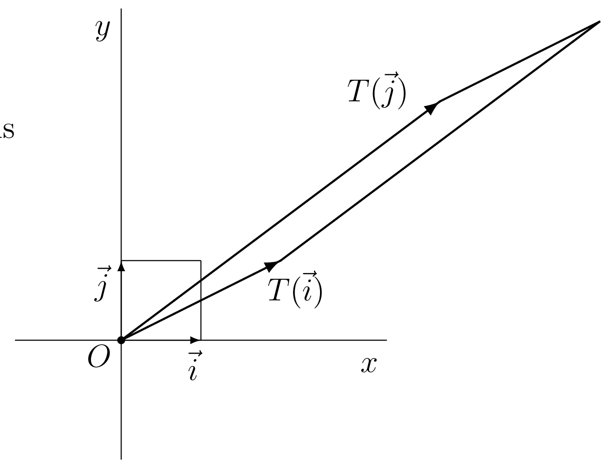
(a) [4 marks] Sketch the image of the unit square.

Solution:

$$T(\vec{\imath}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } T(\vec{\jmath}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

So the image of the unit square is the parallelogram determined by

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$



(b) [4 marks] Find the formula for $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

Solution: The matrix of T is

$$A = [T(\vec{\imath})|T(\vec{\jmath})] = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}.$$

Then the matrix of T^{-1} is

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix},$$

and the formula for T^{-1} is

$$\begin{aligned} T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 3x-4y \\ -x+2y \end{bmatrix} \end{aligned}$$

6. [8 marks] Let $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection on the line with equation $y = -2x$, induced by the matrix A; let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection on the line with equation $y = x/2$, induced by the matrix B. Compute both AB and BA. Interpret your results geometrically.

Solution: a projection on the line $y = mx$ has matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$$

So with $m = -2$,

$$A = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

and with $m = 1/2$,

$$B = \frac{1}{1+1/4}\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

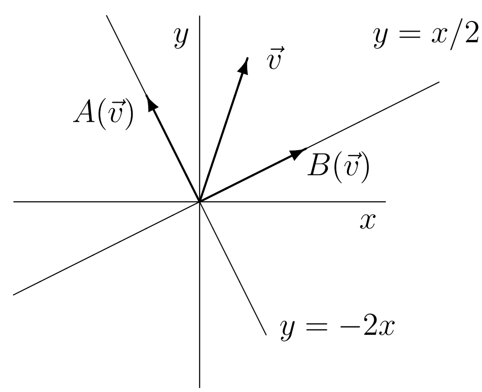
Then

$$AB = \frac{1}{25}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$BA = \frac{1}{25}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Geometric Interpretation:
The lines $y = -2x$ and $y = x/2$ are orthogonal to each other. For any vector $\vec{v}$,
$A(\vec{v})$ is parallel to the line $y = -2x$, and
$B(\vec{v})$ is parallel to the line $y = x/2$.
Thus:
$A(\vec{v})$ is orthogonal to the line $y = x/2$,
and $B(v)$ is orthogonal to the line $y = -2x$.
Hence, for all vectors $\vec{v}$,



$$B(A(\vec{v})) = \vec{0} \text{ and } A(B(\vec{v})) = \vec{0}.$$

This means the two matrices, BA, and AB, are both the zero matrix. Or putting it another way:
Let $\vec{v} \in \mathbb{R}^2$; let $\vec{d_1} = [1 \ -2]^T$ and let $\vec{d_2} = [2 \ 1]^T$. Then $\vec{d_1} \cdot \vec{d_2} = 0$ and

$$(AB)(\vec{v}) = S(T(\vec{v})) = S\left(\operatorname{proj}_{\vec{d_2}}\vec{v}\right) = \operatorname{proj}_{\vec{d_1}}\left(\operatorname{proj}_{\vec{d_2}}\vec{v}\right) = \vec{0},$$

so $AB = O$. Similarly, $BA = O$.

7. [10 marks] Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and determine if A is diagonalizable.

Solution:

$$
\begin{aligned}
& \det(\lambda I - A) \\
= \;& \det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} = \det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ 0 & -1-\lambda & \lambda+1 \end{bmatrix} \\
= \;& (\lambda+1)\det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ 0 & -1 & 1 \end{bmatrix} = (\lambda+1)\det \begin{bmatrix} \lambda & -2 & -1 \\ -1 & \lambda-1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\
= \;& (\lambda+1)\,(\lambda(\lambda-1)-2) = (\lambda+1)(\lambda^2-\lambda-2) \\
= \;& (\lambda+1)^2(\lambda-2)
\end{aligned}
$$

So the **eigenvalues** of A are $\lambda_1 = -1$ and $\lambda_2 = 2$.

Eigenvectors:

$$
(-I-A|O) = \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];\ \text{take } X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.
$$

$$
(2I-A|O) = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];\ \text{take } X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
$$

A **is diagonalizable:** because

$$
P = \left[\begin{array}{c|c|c} X_1 & X_2 & X_3 \end{array}\right] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
$$

is invertible, since $\det(P) = 3 \neq 0$.

8. [9 marks; 3 marks for each part] Indicate if the following statements are True or False, and give a brief explanation why.

(a) If A is a $n \times n$ matrix and $A^T = A^{-1}$, then $\det(A) = \pm 1$. **True** **False**

Solution: True.

$$\begin{aligned} A^T = A^{-1} &\Rightarrow A^T A = I \\ &\Rightarrow \det(A^T A) = \det I \\ &\Rightarrow \det(A^T) \det A = 1 \\ &\Rightarrow (\det(A))^2 = 1 \\ &\Rightarrow \det(A) = \pm 1 \end{aligned}$$

(b) If $\operatorname{adj}(A)$ is the 3×3 zero matrix then so is A. **True** **False**

Solution: False. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix};$$

then $\operatorname{adj}(A) = O$, but $A \neq O$.

(c) If $\lambda = 0$ is an eigenvalue of A then A is invertible. **True** **False**

Solution: False. Let $X \neq O$ be an eigenvector of A. Then

$$AX = 0X = O \Rightarrow A \text{ is not invertible.}$$

OR: if $\lambda = 0$ is an eigenvalue of A then

$$\det(0I - A) = 0 \Rightarrow \det(-A) = 0 \Rightarrow (-1)^n \det(A) = 0 \Rightarrow \det(A) = 0,$$

which means A is not invertible.

OR: give one counter example. The zero matrix is not invertible but has eigenvalue 0.