# MAT188H1F - Linear Algebra - Fall 2014

## Solutions to Term Test 2 - November 6, 2014

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

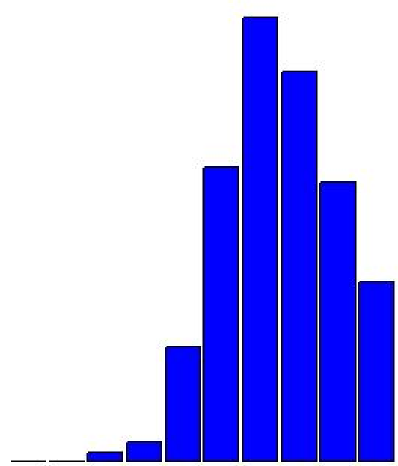
This test consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. Questions 1, 2, 3, 4 and 8 were quite well done, for the most part.
2. Many students had difficulties with Questions 5, 6 and 7. This was mainly because each of these questions required some explanation. When you explain something mathematically you should present a clear, coherent argument that can be read from start to finish and makes sense. Instead, many students wrote down "floating expressions" with no indication of what they were doing, or why they were doing it, or how they are all connected! This will not get you many marks, if any.

**Breakdown of Results:** 947 students wrote this test. The marks ranged from 23.4% to 100%, and the average was 69.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | % | Decade | % |
|---|---|---|---|
| | | 90-100% | 10.3% |
| A | 26.5% | 80-89% | 16.2% |
| B | 22.5% | 70-79% | 22.5% |
| C | 25.7% | 60-69% | 25.7% |
| D | 17.0% | 50-59% | 17.0% |
| F | 8.3% | 40-49% | 6.6% |
| | | 30-39% | 1.2% |
| | | 20-29% | 0.5% |
| | | 10-19% | 0.0% |
| | | 0-9% | 0.0% |



## PART I : No explanation is necessary.

1. (avg: 9.4/10) Given that the reduced echelon form of

$$A = \begin{bmatrix} 1 & -2 & -1 & 1 & -2 \\ 4 & -8 & -4 & 5 & -7 \\ 2 & -4 & -2 & 3 & -3 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -2 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. Put your answers in the blanks to the right.

(a) [1 mark] dim(row($A$)) $\underline{\quad 2 \quad}$

(b) [1 mark] dim(col($A$)) $\underline{\quad 2 \quad}$

(c) [1 mark] dim(null($A$)) $\underline{\quad 3 \quad}$

(d) [2 marks] A basis for the row space of $A$

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(e) [2 marks] A basis for the column space of $A$

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$$

(f) [3 marks] A basis for the null space of $A$

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

2. (avg: 8.3/10) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ and $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be linear transformations defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ 3x_1 + 2x_2 \\ -x_1 + x_2 \end{bmatrix}, \quad S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 - x_4 \\ -x_1 + x_2 + x_3 + 2x_4 \end{bmatrix}.$$

Put your answers to the following questions in the blanks to the right.

(a) [1 mark] What is the matrix of $T$? $\begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}$

(b) [1 mark] What is the matrix of $S$? $\begin{bmatrix} 1 & -2 & 1 & -1 \\ -1 & 1 & 1 & 2 \end{bmatrix}$

(c) [1 mark] Is $T$ one-to-one? Yes

(d) [1 mark] Is $T$ onto? No

(e) [1 mark] Is $S$ one-to-one? No

(f) [1 mark] Is $S$ onto? Yes

(g) [2 marks] Is $S \circ T$ one-to-one, where $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$? Yes

The matrix of $S \circ T$ is $\begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix}$, which is invertible.

(h) [2 marks] Is $T \circ S$ onto, where $(T \circ S)(\mathbf{x}) = T(S(\mathbf{x}))$? No

The matrix of $T \circ S$ is $\begin{bmatrix} 3 & -5 & 1 & -4 \\ 0 & -1 & 2 & 1 \\ 1 & -4 & 5 & 1 \\ -2 & 3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 1 & -4 \\ 0 & -1 & 2 & 1 \\ 0 & 7 & -14 & -7 \\ 0 & -5 & 10 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 1 & -4 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.



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**PART II :** Present **COMPLETE** solutions to the following questions in the space provided.

3. (avg: 8.4/10) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x-y \end{bmatrix}.$$

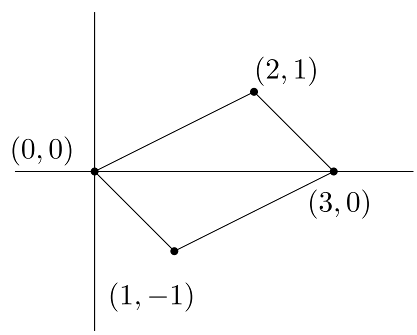
(a) [5 marks] Draw the image of the unit square[1] under $T$ and label all four vertices.

**Solution:** the image of the unit square is the parallelogram determined by

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$




(b) [5 marks] Find $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

**Solution:** if the matrix of $T$ is $A$ then the matrix of $T^{-1}$ is $A^{-1}$. We have

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}; \quad A^{-1} = \frac{1}{-2-1}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

So

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x+y \\ x-2y \end{bmatrix}.$$

[1] The unit square is the square with the four vertices $(0,0), (1,0), (0,1), (1,1)$.

4. (avg: 9.0/10) Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 7 \\ -1 & -1 & 5 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

Find $A^{-1}$ and use it to solve the equation $A\,\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

**Solution:** use the Gaussian algorithm.

$$(A|I) = \left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 7 & 0 & 1 & 0 \\ -1 & -1 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 2 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & 13 & 4 & -3 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & -28 & -9 & 7 \\ 0 & 1 & 0 & 13 & 4 & -3 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right] = (I|A^{-1})$$

So

$$A^{-1} = \begin{bmatrix} -28 & -9 & 7 \\ 13 & 4 & -3 \\ -3 & -1 & 1 \end{bmatrix}$$

and

$$\mathbf{x} = A^{-1}\,\mathbf{b} = \begin{bmatrix} -28 & -9 & 7 \\ 13 & 4 & -3 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 40 \\ -19 \\ 4 \end{bmatrix}.$$

5. (avg: 4.1/10) Let $S$ be the subset of $\mathbb{R}^4$ consisting of vectors of the form $\begin{bmatrix} a-2c \\ a+b-2c \\ -2a+4c \\ 3b \end{bmatrix}$, where $a, b, c$ are real numbers.

(a) [6 marks] Show that $S$ is a subspace of $\mathbb{R}^4$. (It is *not* necessary to use the definition of subspace.)

**Solution:** show $S$ is the span of a set of vectors, then it must be a subspace.

$$
\begin{aligned}
S = \left\{ \begin{bmatrix} a-2c \\ a+b-2c \\ -2a+4c \\ 3b \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} &= \left\{ a \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} -2 \\ -2 \\ 4 \\ 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \\
&= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 4 \\ 0 \end{bmatrix} \right\}
\end{aligned}
$$

(b) [4 marks] Find a basis for $S$ and its dimension.

**Solution:** the above spanning set contains a basis. Since the third vector above is $-2$ times the first vector, and the second vector is not a multiple of the first, a basis for $S$ is

$$
\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}
$$

and $\dim(S) = 2$.

6. (avg: 6.7/10) Let $A$ be a $5 \times 7$ matrix.

   (a) [5 marks] Could $\dim(\text{null}(A)) = 1$? Make sure to justify your answer.

   **Solution:** use the rank-nullity theorem. Let $r$ be the rank of $A$. Since $r$ is both the number of independent rows of $A$ and the number of independent columns of $A$ we must have $r \leq 5$. Then

$$\dim(\text{null}(A)) = 7 - r \geq 2,$$

   and $\dim(\text{null}(A)) \neq 1$.

   (b) [5 marks] Suppose $\dim(\text{null}(A)) = 3$. Find $\dim(\text{null}(A^T))$.

   **Solution:** use the rank-nullity theorem. First, since $\dim(\text{null}(A)) = 3$ the rank of $A$ is

$$r = 7 - 3 = 4.$$

   $A^T$ is $7 \times 5$. Also, $A$ and $A^T$ have the same rank. Thus

$$\dim(\text{null}(A^T)) = 5 - \text{rank}(A^T) = 5 - 4 = 1.$$



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7. (avg: 3.4/10) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$. Find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that

$$\ker(T) = \operatorname{span}\{\mathbf{v}, \mathbf{w}\} \text{ and } \operatorname{range}(T) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}.$$

**Solution:** there are many ways to do this problem. One way is let the matrix of $T$ be

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Since $\operatorname{range}(T) = \operatorname{col}(A)$ we have

$$\begin{bmatrix} a \\ d \end{bmatrix}, \begin{bmatrix} b \\ e \end{bmatrix}, \begin{bmatrix} c \\ f \end{bmatrix} \text{ are all in span} \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\};$$

that is, $a = b = c = 0$.

To find $d, e, f$ use the fact that $\ker(T) = \operatorname{null}(A)$. Hence $A\,\mathbf{v} = \mathbf{0}$ and $A\,\mathbf{w} = \mathbf{0}$. This gives us two equations for $d, e, f$:

$$\left\{\begin{array}{rcrcrcl} d & + & e & + & f & = & 0 \\ -d & & & - & f & = & 0 \end{array}\right.,$$

with solutions $e = 0$ and $f = -d$. So any matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ d & 0 & -d \end{bmatrix}, d \neq 0$$

will do. In particular, could take $d = 1$ and have

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 - x_3 \end{bmatrix}.$$

8. (avg; 6.2/10) Indicate if the following statements are **True** or **False**, and give a *brief* explanation why.

(a) [2 marks] If $A$ is a $2 \times 2$ matrix such that $A^2 = I$, then $A = \pm I$. **False**

**Solution:** consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A^2 = I$ but $A \neq \pm I$.

(b) [2 marks] If $A$ is an $n \times m$ matrix then $AA^T$ is symmetric. **True**

**Solution:**

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

(c) [2 marks] $\dim \left( \operatorname{span} \left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} \right\} \right) = 3$ **True**

**Solution:** the three given vectors form a basis of $\mathbb{R}^3$ since they are independent:

$$\begin{bmatrix} 4 & 1 & 8 \\ 3 & 0 & 6 \\ 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & -3 & -9 \\ 0 & -3 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & -3 & -9 \\ 0 & 0 & -3 \end{bmatrix}$$

(d) [2 marks] $\operatorname{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 2 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 8 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^4$ **False**

**Solution:** the second vector is twice the first, and the last is eight times the fourth, so there are only three independent vectors in the spanning set—and three vectors cannot span $\mathbb{R}^4$.

(e) [2 marks] If $A$ is an $n \times n$ matrix such that $A^2 = I$, then $\operatorname{null}(A) = \{\mathbf{0}\}$. **True**

**Solution:** $AA = I$ means $A$ is invertible (and $A^{-1} = A$) which means $\operatorname{null}(A) = \{\mathbf{0}\}$.

This page is for rough work; it will not be marked.