MAT188H1F - Linear Algebra - Fall 2015

Solutions to Term Test 2 - November 17, 2015

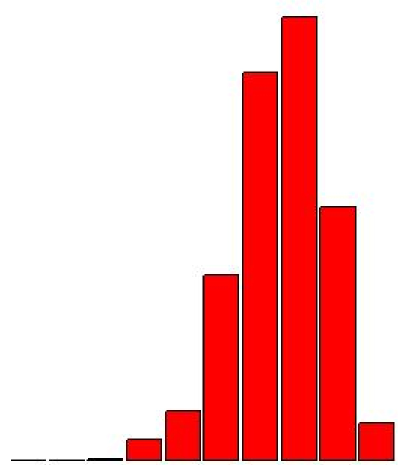
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

1. The results on this test were–not surprisingly–significantly lower than on the first test. Nevertheless the failure rate was still very low. On the other hand, there were only 25 students with A^+.
2. The only two questions with a failing average were Questions 5 (Subspace) and 8 (True or False). The range on every question was 0 to 10; except for Question 8 which had a range of 0 to 9.
3. Counting the first test as 15% and the second test as 20%, the average mark out of 35 is presently 26.5, or 75.7%.

Breakdown of Results: 927 students wrote this test. The marks ranged from 21.25% to 96.25%, and the average was 69.69%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.7%
A	21.0%	80-89%	18.3 %
B	32.1%	70-79%	32.1%
C	28.2%	60-69%	28.2%
D	13.5 %	50-59%	13.5%
F	5.2%	40-49%	3.6%
		30-39%	1.5%
		20-29%	0.1%
		10-19%	0.0%
		0-9%	0.0%



PART I : No explanation is necessary.

1. [avg: 9.37/10] Given that the reduced echelon form of

$$A = \begin{bmatrix} 1 & -2 & -1 & 1 & -2 \\ 4 & -8 & -4 & 5 & -7 \\ 2 & -4 & -2 & 3 & -3 \\ -1 & 2 & 2 & 1 & 1 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -2 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. Put your answers in the blanks to the right. Write all your vectors as columns!

(a) [1 mark] rank(A).

3

(b) [3 marks] A basis for the row space of A.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(c) [3 marks] A basis for the column space of A.

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \\ 1 \end{bmatrix} \right\}$$

(d) [3 marks] A basis for the null space of A.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Continued...

PART II : Present **COMPLETE** solutions to the following questions in the space provided.

2. [avg: 8.13/10] For any real numbers a, b, c, let $A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

   (a) [3 marks] Find $\det(A)$. Is A invertible?

   Soluton: use the formula for the determinant of a 3×3 matrix:

   $$\det(A) = 1 - abc + abc + c^2 + b^2 + a^2 = 1 + a^2 + b^2 + c^2 \geq 1.$$

   So $\det(A) \neq 0$ and A is invertible.

   (b) [3 marks] Find $\det(B)$. Is B invertible?

   Solution: similarly,

   $$\det(B) = 0 - abc + abc + 0 + 0 + 0 = 0.$$

   So B is not invertible.

   (c) [4 marks] If either A or B is invertible, find its inverse.

   Soluton: use the adjoint formula.

   $$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) \\ &= \frac{1}{1 + a^2 + b^2 + c^2} \begin{bmatrix} 1 + c^2 & -(-a + bc) & ac + b \\ -(a + bc) & 1 + b^2 & -(-c + ab) \\ ac - b & -(c + ab) & 1 + a^2 \end{bmatrix}^T \\ &= \frac{1}{1 + a^2 + b^2 + c^2} \begin{bmatrix} 1 + c^2 & -a - bc & -b + ac \\ a - bc & 1 + b^2 & -c - ab \\ b + ac & c - ab & 1 + a^2 \end{bmatrix} \end{aligned}$$

3. [avg: 8.98/10] Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ 5x - 2y \end{bmatrix}.$$

(a) [5 marks] Draw the image of the unit square[1] under T, label all of its vertices, and find its area.

Solution: the image of the unit square is the parallelogram determined by

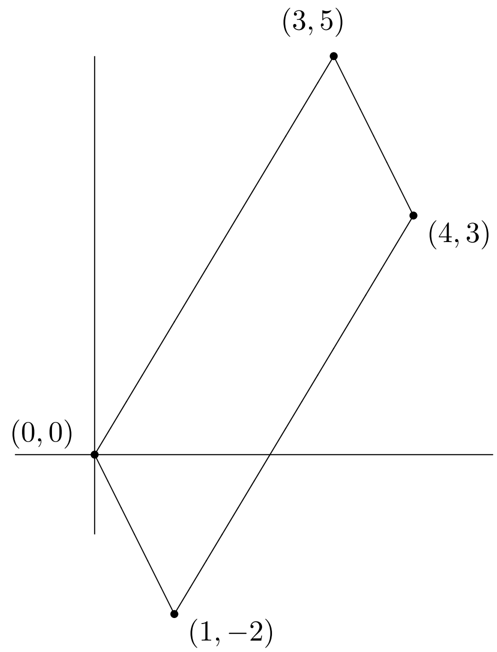
$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

and

$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Its area is given by

$$\left|\det \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix}\right| = |-6-5| = 11.$$



(b) [5 marks] Find $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

Solution: if the matrix of T is A then the matrix of T^{-1} is A^{-1}. We have

$$A = \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix}; \quad A^{-1} = \frac{1}{-6-5}\begin{bmatrix} -2 & -1 \\ -5 & 3 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}.$$

So

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{11}\begin{bmatrix} 2 & 1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 2x + y \\ 5x - 3y \end{bmatrix}.$$

[1] The unit square is the square with the four vertices $(0,0), (1,0), (0,1), (1,1)$.

4. [avg: 6.57/10] Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 4 & 1 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) [4 marks] Verify that $A^{-1} = \begin{bmatrix} -1 & 6 & -1 \\ 0 & -4 & 1 \\ 1 & -5 & 1 \end{bmatrix}$.

Solution: check that $A^{-1}A = I$ (or $AA^{-1} = I$):

$$\begin{bmatrix} -1 & 6 & -1 \\ 0 & -4 & 1 \\ 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 4 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -1+6-4 & 1+0-1 & -2+6-4 \\ 0-4+4 & 0+0+1 & 0-4+4 \\ 1+5-4 & -1+0+1 & 2-5+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) [2 marks] Use A^{-1} to solve the equation $A\,\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

Solution:

$$\mathbf{x} = A^{-1}\,\mathbf{b} = \begin{bmatrix} -1 & 6 & -1 \\ 0 & -4 & 1 \\ 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

(c) [4 marks] Use A^{-1} to solve the equation $A^T A\,\mathbf{y} = \mathbf{b}$ for $\mathbf{b}$.

Solution: $\mathbf{y} = (A^T A)^{-1}\mathbf{b} = A^{-1}(A^T)^{-1}\,\mathbf{b} = A^{-1}(A^{-1})^T\,\mathbf{b}$. That is,

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} -1 & 6 & -1 \\ 0 & -4 & 1 \\ 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 6 & -1 \\ 0 & -4 & 1 \\ 1 & -5 & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 6 & -1 \\ 0 & -4 & 1 \\ 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 6 & -4 & -5 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 6 & -1 \\ 0 & -4 & 1 \\ 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -5 \end{bmatrix} \end{aligned}$$

5. [avg: 4.46/10] Let $T : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a linear transformation, and let S be the subset of $\mathbf{R}^n$ consisting of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = 3\,\mathbf{x}$.

(a) [4 marks] Show that S is a subspace of $\mathbf{R}^n$.

Solution; let A be the matrix of T. Then $T(\mathbf{x}) = 3\,\mathbf{x} \Leftrightarrow A\,\mathbf{x} = 3\,\mathbf{x} \Leftrightarrow (A - 3I)\,\mathbf{x} = \mathbf{0}$.

So $S = \text{null}(A - 3I)$ and is necessarily a subspace of $\mathbf{R}^n$.

OR use the subspace test:

1. $\mathbf{0}$ is in S: $T(\mathbf{0}) = \mathbf{0} = 3\,\mathbf{0}$
2. S is closed under addition: let $\mathbf{x}, \mathbf{y}$ be in S.

$$T(\mathbf{x}) = 3\,\mathbf{x} \text{ and } T(\mathbf{y}) = 3\,\mathbf{y} \Rightarrow T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y}) = 3\,\mathbf{x}+3\,\mathbf{y} = 3(\mathbf{x}+\mathbf{y}) \Rightarrow \mathbf{x}+\mathbf{y} \in S$$

3. S is closed under scalar multiplication: let $\mathbf{x}$ be in S, k a scalar in $\mathbf{R}$.

$$T(\mathbf{x}) = 3\,\mathbf{x} \Rightarrow T(k\mathbf{x}) = kT(\mathbf{x}) = k(3\mathbf{x}) = 3(k\mathbf{x}) \Rightarrow k\mathbf{x} \in S$$

(b) [2 marks] Find an example of T such that $\dim(S) = n$.

Solution: take $T(\mathbf{x}) = 3\,\mathbf{x}$, for all $\mathbf{x}$. Then $S = \mathbf{R}^n$.

(c) [2 marks] Find an example of T such that $\dim(S) = 0$.

Solution: could take $T(\mathbf{x}) = \mathbf{x}$, for all $\mathbf{x}$; or $T(\mathbf{x}) = \mathbf{0}$, for all $\mathbf{x}$. Either way, $\mathbf{x} = \mathbf{0}$ is the only vector that satisfies $T(\mathbf{x}) = 3\,\mathbf{x}$. That is, $S = \{\mathbf{0}\}$.

(d) [HARD 2 marks] Show that if $\dim(\text{range}(T)) = 1$ and $S \neq \{\mathbf{0}\}$, then $\text{range}(T) = S$.

Solution: you need to make two observations:

1. S is contained in the range of T:

$$\mathbf{x} \text{ in } S \Rightarrow T(\mathbf{x}) = 3\,\mathbf{x} \text{ is in range}(T) \Rightarrow \frac{1}{3}(3\,\mathbf{x}) = \mathbf{x} \text{ is in range}(T)$$

2. $\dim(S) = \dim(\text{range}(T))$:

$$0 < \dim(S) \leq \dim(\text{range}(T)) = 1$$

$$\Rightarrow \dim(S) = 1$$

Consequently, $S = \text{range}(T)$.

6. [avg: 6.75/10] Suppose A is a 15×10 matrix with rank equal to 7.

(a) [4 marks] Let T be the linear transformation defined by $T(\mathbf{x}) = A\,\mathbf{x}$, for all $\mathbf{x}$ in $\mathbf{R}^{10}$. What is the dimension of the kernel of T?

Solution: $\dim(\ker(T)) = \text{nullity}(A) = 10 - \text{rank}(A) = 10 - 7 = 3$.

(b) [6 marks; 2 for each part] Let S be the linear transformation defined by $S(\mathbf{y}) = A^T\,\mathbf{y}$, for all $\mathbf{y}$ in $\mathbf{R}^{15}$. Determine if the following statements could be true.

Note: A^T is a 10×15 matrix, so $S : \mathbf{R}^{15} \longrightarrow \mathbf{R}^{10}$.

(*i*) S is onto.

Solution: NOT possible, since $\dim(\text{range}(S)) = \text{rank}(A^T) = \text{rank}(A) = 7 < 10$.

(*ii*) S is one-to-one.

Solution: NOT possible,
since $\dim(\ker(S)) = \text{nullity}(A^T) = 15 - \text{rank}(A^T) = 15 - 7 = 8 > 0$.

(*iii*) $S \circ T$ is onto, with T as in part (a) above. Recall: $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$.

Solution: NOT possible. Since $S \circ T : \mathbf{R}^{10} \longrightarrow \mathbf{R}^{10}$, $S \circ T$ is onto if and only if it is one-to-one, by the Big Theorem. But $S \circ T$ can't be one-to-one because T isn't, by part (a). That is, there is a non-zero vector $\mathbf{x} \in \mathbf{R}^{10}$ such that $T(\mathbf{x}) = \mathbf{0}$. Then $\mathbf{x} \neq \mathbf{0}$, and

$$(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = S(\mathbf{0}) = \mathbf{0},$$

which means $\ker(S \circ T) \neq \{\mathbf{0}\}$.

7. [avg: 6.74/10] Find all values of the parameter a for which the system of equations

$$\begin{cases} x_1 + a\,x_2 + x_3 = 2 \\ 3x_1 - x_2 + a\,x_3 = -4 \\ x_1 + 2x_2 + x_3 = 1 \end{cases}$$

has *(i)* no solution, *(ii)* a unique solution, *(iii)* infinitely many solutions.

Solution: let the coefficient matrix be $A = \begin{bmatrix} 1 & a & 1 \\ 3 & -1 & a \\ 1 & 2 & 1 \end{bmatrix}$.

$$\det(A) = -1 + a^2 + 6 + 1 - 2a - 3a = a^2 - 5a + 6 = (a-2)(a-3).$$

If the coefficient matrix A is invertible, the system of equations will have a unique solution. Therefore, *(ii)*, the system has a unique solution if $\det(A) \neq 0 \Leftrightarrow a \neq 2, a \neq 3$.

Now see what happens for $a = 2$ or 3:

(i): if $a = 2$ the system has no solution, since the first and third equations are inconsistent:

$$\begin{cases} x_1 + 2\,x_2 + x_3 = 2 \\ 3x_1 - x_2 + 2\,x_3 = -4 \\ x_1 + 2x_2 + x_3 = 1 \end{cases}$$

(iii): for $a = 3$ the system has infinitely many solutions, $x_1 = -1 - t, x_2 = 1, x_3 = t$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 3 & -1 & 3 & -4 \\ 1 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 10 & 0 & 10 \\ 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Alternate Soluiton: find an echelon form of the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 2 \\ 3 & -1 & a & -4 \\ 1 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & a & 1 & 2 \\ 0 & -1-3a & a-3 & -10 \\ 0 & 2-a & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & a & 1 & 2 \\ 0 & 2-a & 0 & -1 \\ 0 & 0 & (2-a)(a-3) & 7a-21 \end{array}\right]$$

and analyze the cases when $a \neq 2, 3$, or $a = 2$; or $a = 3$.

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8. [avg: 4.75/10] Indicate if the following statements are **True** or **False**, and give a *brief* explanation why.

(a) [2 marks] If A is a 3×2 matrix and B is a 2×3 matrix, then $\det(AB) = \det(BA)$.

True $\boxed{\textbf{False}}$

Explanation: consider $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $A = B^T$. Then

$$BA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \det(BA) = 2 \text{ and } \det(AB) = 0.$$

(b) [2 marks] Let A be an $n \times n$ matrix with n pivot columns. Suppose $AB = AC$. Then $B = C$.

$\boxed{\textbf{True}}$ **False**

Explanation: A is invertible, so $A^{-1}AB = A^{-1}AC \Rightarrow B = C$.

(c) [2 marks] If A is an $n \times n$ non-zero matrix such that $A^T = kA$, then $k = \pm 1$. $\boxed{\textbf{True}}$ **False**

Explanation: $A^T = kA \Rightarrow (A^T)^T = (kA)^T \Rightarrow A = kA^T = k(kA) = k^2 A$.
But $A \neq 0$, so $k^2 = 1$.

(d) [2 marks] If $\left\{ \begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix}^T, \begin{bmatrix} 3 & 1 & 2 & 4 \end{bmatrix}^T, \begin{bmatrix} 0 & 2 & 3 & 1 \end{bmatrix}^T \right\}$ is a basis for S, then $\left\{ \begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix}^T, \begin{bmatrix} 3 & 1 & 2 & 4 \end{bmatrix}^T, \begin{bmatrix} 4 & 3 & 7 & 4 \end{bmatrix}^T \right\}$ is also a basis for S.

$\boxed{\textbf{True}}$ **False**

Explanation: both sets have three vectors and they are both spanning sets of S, since

$$\begin{bmatrix} 4 & 3 & 7 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 2 & -1 \end{bmatrix}^T + \begin{bmatrix} 3 & 1 & 2 & 4 \end{bmatrix}^T + \begin{bmatrix} 0 & 2 & 3 & 1 \end{bmatrix}^T$$

(e) [2 marks] For any real number k and any $n \times n$ matrix A, $\det(kA) = k \det(A)$.

True $\boxed{\textbf{False}}$

Explanation: $\det(kA) = k^n \det(A)$, so the above formula is not true if $k \neq 0, k \neq 1$.

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