MAT188H1F - Linear Algebra - Fall 2018

Solutions to Term Test 2 - November 6, 2018

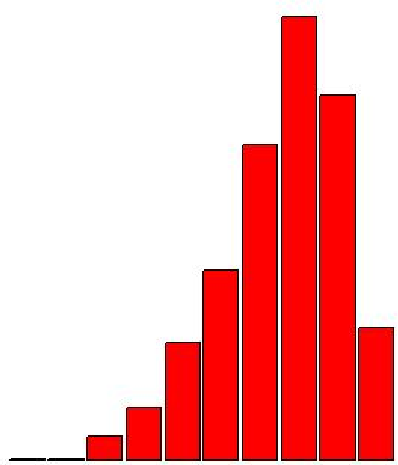
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

- In Question 2(b), 111 students thought the volume was negative.
- On the whole, the proofs in Question 7 were not well done; especially considering that part (a) was Question 5 in the Tutorial 7 handout, and that part (b) could be done using the same method as Question 4 in the Tutorial 7 handout. For that matter, Question 5(c) was a special case of Question 3 on the Tutorial 7 handout.
- In Question 6(b) a common mistake was to use an arbitrary vector $\vec{n}$ that is orthogonal to both lines; in fact, there is only *one* normal vector that can be used, namely the vector that is orthogonal to the normal vector of the plane that contains both lines, and is orthogonal to the lines. (See Alternate Solutions on page 11.)

Breakdown of Results: 823 registered students wrote this test. The marks ranged from 7.5% to 100%, and the average was 69.8%. There was 1 perfect paper. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	8.0%
A	30.2%	80-89%	22.2%
B	27.0%	70-79%	27.0%
C	19.2%	60-69%	19.2%
D	11.5%	50-59%	11.5%
F	12.1%	40-49%	7.2%
		30-39%	3.2%
		20-29%	1.5%
		10-19%	0.1%
		0-9%	0.1%



1.[avg: 9.19/10] Let $\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}$. Calculate the following:

(a) [2 marks] $\|\vec{u}\|$

Solution: $\|\vec{u}\| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}$.

(b) [2 marks] $\|\vec{v}\|$

Solution: $\|\vec{v}\| = \sqrt{6^2 + 3^2 + (-3)^2} = \sqrt{54}$.

(c) [2 marks] $\vec{u} \cdot \vec{v}$

Solution: $\vec{u} \cdot \vec{v} = (-1)(6) + (1)(3) + (2)(-3) = -9$.

(d) [2 marks] The angle θ between the vectors $\vec{u}$ and $\vec{v}$.

Solution:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = -\frac{9}{\sqrt{6}\sqrt{54}} = -\frac{1}{2},$$

so

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

(e) [2 marks] $\mathrm{proj}_{\vec{v}}\,(\vec{u})$.

Solution:

$$\mathrm{proj}_{\vec{v}}\,(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = -\frac{9}{54}\vec{v} = -\frac{1}{6}\vec{v}.$$

2. [avg: 8.18/10]

2.(a) [5 marks] Find the area of the triangle with vertices $P(1,1,3), Q(4,3,5), R(2,6,4)$

Solution: let A be the area of the triangle. Then

$$A = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\left\| \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \right\| = \frac{1}{2}\left\| \begin{bmatrix} -8 \\ -1 \\ 13 \end{bmatrix} \right\| = \frac{1}{2}\sqrt{64+1+169} = \frac{\sqrt{234}}{2}$$

Of course, either of

$$\frac{1}{2}\|\overrightarrow{QP} \times \overrightarrow{QR}\| \text{ or } \frac{1}{2}\|\overrightarrow{RP} \times \overrightarrow{RQ}\|$$

will work equally well.

2.(b) [5 marks] Find the volume of the parallelepiped determined by the three vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Solution: let V be the volume of the parallelepiped. Then

$$\begin{aligned} V = |\det[\vec{u}\ \vec{v}\ \vec{w}]| &= \left| \det \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right| \\ &= \left| \det \begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right| \\ &= \left| \det \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \right| \\ &= |-3| = 3 \end{aligned}$$

Of course, you can also use triple products and say that $V = |\vec{u} \cdot \vec{v} \times \vec{w}|$; it's the same, since

$$\vec{u} \cdot \vec{v} \times \vec{w} = \det[\vec{u}\ \vec{v}\ \vec{w}].$$

3. [avg: 9.0/10] Consider the system of equations $(*)\ \left\{\begin{array}{rcrcrcr} 7x_1 & + & 2x_2 & - & 6x_3 & = & 2 \\ -3x_1 & - & x_2 & + & 3x_3 & = & -4 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 6 \end{array}\right.$.

(a) [3 marks] Write this system as a single matrix equation $A\,\vec{x} = \vec{b}$. Clearly identify $A, \vec{x}$ and $\vec{b}$.

Solution:

$$\underbrace{\begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}}_{\vec{b}}$$

(b) [4 marks] Find A^{-1}.

Solution: use the Gaussian algorithm to find A^{-1} :

$$[A|I] = \left[\begin{array}{rrr|rrr} 7 & 2 & -6 & 1 & 0 & 0 \\ -3 & -1 & 3 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 0 \\ -3 & -1 & 3 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & 3 & 3 & 7 & 0 \\ 0 & 1 & -2 & -2 & -4 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & 3 & 3 & 7 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & -3 & -3 & -7 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right] = [I|A^{-1}]$$

That is,

$$A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}.$$

(c) [3 marks] Use A^{-1} to solve the system of equations $(*)$.

Solution: $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \\ -4 \end{bmatrix}$$

4. [avg: 7.98/10] Solve the following system of linear equations

$$\begin{array}{rcrcr} f_1' & = & 2f_1 & + & 4f_2 \\ f_2' & = & 3f_1 & + & 3f_2 \end{array}$$

for f_1 and f_2 as functions of x if $f_1(0) = 0$ and $f_2(0) = 1$.

Solution: the coefficient matrix is $A = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$, for which

$$\det(\lambda I - A) = (\lambda - 2)(\lambda - 3) - 12 = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1).$$

The eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = -1$, the roots of the characteristic polynomial. To find the eigenvectors, find a (simple) non-zero solution $\vec{v}$ to the homogeneous system $(\lambda I - A)\vec{v} = \vec{0}$:

For eigenvalue $\lambda_1 = 6$:

$$\left[\begin{array}{rr|r} 4 & -4 & 0 \\ -3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right];$$

take

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For eigenvalue $\lambda_2 = -1$:

$$\left[\begin{array}{rr|r} -3 & -4 & 0 \\ -3 & -4 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right];$$

take

$$\vec{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

So the general solution is

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6x} + c_2 \begin{bmatrix} -4 \\ 3 \end{bmatrix} e^{-x}.$$

Use the initial conditions at $x = 0$ to find c_1, c_2 :

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Thus

$$f_1(x) = \frac{4}{7} e^{6x} - \frac{4}{7} e^{-x} \text{ and } f_2(x) = \frac{4}{7} e^{6x} + \frac{3}{7} e^{-x}.$$

5. [avg; 6.37/10] Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the reflection in a line defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = -\frac{1}{5}\begin{bmatrix} 4x+3y \\ 3x-4y \end{bmatrix}$$

(a) [2 mark] Find A, the matrix of T.

Solution:

$$A = -\frac{1}{5}\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}.$$

(b) [6 marks] Find the eigenvalues and eigenvectors of A.

Solution: the characteristic polynomial of A is

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda + \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \lambda - \frac{4}{5} \end{bmatrix} = \lambda^2 - \frac{16}{25} - \frac{9}{25} = \lambda^2 - 1,$$

so the eigenvalues of A are $\lambda = \pm 1$, the roots of the characteristic polynomial. To find the eigenvectors, find a (simple) non-zero solution $\vec{v}$ to the homogeneous system $(\lambda I - A)\vec{v} = \vec{0}$:

For eigenvalue $\lambda_1 = 1$:

$$\left[\begin{array}{cc|c} \frac{9}{5} & \frac{3}{5} & 0 \\ \frac{3}{5} & \frac{1}{5} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right];$$

take

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

For eigenvalue $\lambda_2 = -1$:

$$\left[\begin{array}{cc|c} -\frac{1}{5} & \frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{9}{5} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right];$$

take

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

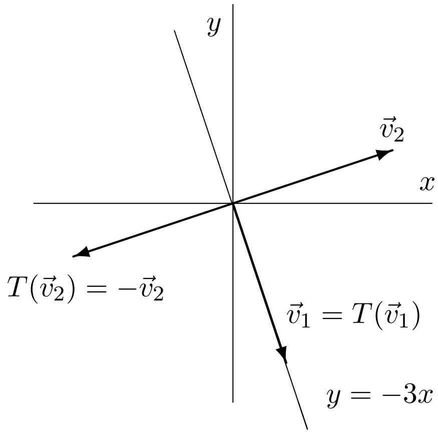
(c) [2 marks] Explain your results from part (b) geometrically in terms of the transformation T.

Solution: $T = Q_{-3}$, where Q_m is the reflection in the line $y = mx$. That is, the line with equation $y = -3x$ is the axis of reflection, and $\vec{v}_1$ is parallel to that line, so

$$T(\vec{v}_1) = \vec{v}_1.$$

On the other hand, $\vec{v}_2$ is orthogonal to $\vec{v}_1$, since $\vec{v}_1 \cdot \vec{v}_2 = 0$, so T flips the direction of $\vec{v}_2$:

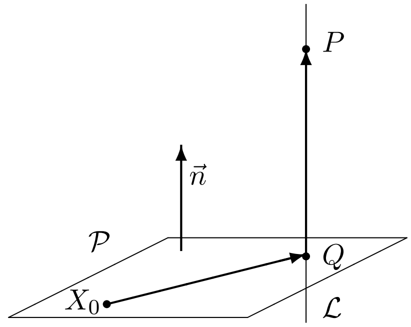
$$T(\vec{v}_2) = -\vec{v}_2.$$



6. [avg: 6.39/10]

6.(a) [5 marks] Find the minimum distance from the point $P(2,3,0)$ to the plane $2x+y-z=6$.

Solution: let $\mathcal{P}$ be the plane with equation $2x+y-z=6$.



- A normal vector to $\mathcal{P}$ is $\vec{n} = [2\ 1\ -1]^T$.
- $X_0(3,0,0)$ is a point on $\mathcal{P}$.
- Then the minimum distance from P to the plane is given by

$$D = \|\text{proj}_{\vec{n}}\left(\overrightarrow{X_0P}\right)\| = \frac{|\overrightarrow{X_0P}\cdot\vec{n}|}{\|\vec{n}\|}.$$

We have,

$$\overrightarrow{X_0P} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \ \vec{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \text{ and so } D = \frac{|-2+3-0|}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}}.$$

6.(b) [5 marks] Find the minimum distance between the two parallel lines with vector equations

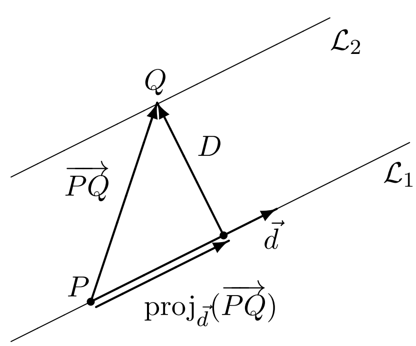
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Solution: since the two lines are parallel you can just find the minimum distance from *any* point on one line to the other line. Let $\mathcal{L}_1$ be the line through the point $P(3,3,5)$; let $\mathcal{L}_2$ be the line through the point $Q(-1,2,5)$. For both lines we can take direction vector $\vec{d} = [2\ -1\ 3]^T$. Then the minimum distance D from Q to $\mathcal{L}_1$ is given by $D = \|\overrightarrow{PQ} - \text{proj}_{\vec{d}}(\overrightarrow{PQ})\|$

$$= \left\|\begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix} - \frac{\overrightarrow{PQ}\cdot\vec{d}}{\|\vec{d}\|^2}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\|$$

$$= \left\|\begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\|$$

$$= \frac{1}{2}\left\|\begin{bmatrix} -6 \\ -3 \\ 3 \end{bmatrix}\right\| = \frac{3\sqrt{6}}{2}$$



7. [avg: 2.48/10]

7.(a) [5 marks] Prove that the $n \times n$ matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Solution: there are many ways to do this. Recall that by definition, $\lambda = 0$ is an eigenvalue of A if there is a **non-zero** vector $\vec{v}$ such that $A\vec{v} = 0\,\vec{v} = \vec{0}$.

Proof 1: A is invertible

$\Leftrightarrow$ the homogenous system of equations $A\vec{x} = \vec{0} = 0\,\vec{x}$ has only the trivial solution

$\Leftrightarrow$ $\lambda = 0$ is not an eigenvalue of A.

Proof 2: the given statement is logically equivalent to the statement

A is not invertible if and only if $\lambda = 0$ is an eigenvalue of A.

We have:

A is not invertible

$\Leftrightarrow \det(A) = 0$

$\Leftrightarrow \det(-A) = 0$

$\Leftrightarrow \det(0I - A) = 0$

$\Leftrightarrow \lambda = 0$ is an eigenvalue of A.

7.(b) [5 marks] If P is the matrix of *any* projection transformation, be it onto a line or onto a plane, then $P^2 = P$. Show that the only eigenvalues of P are $\lambda = 0$ or $\lambda = 1$. (Hint: let $\vec{v}$ be an eigenvector of P.)

Solution: let $\vec{v}$ be an eigenvector of P with eigenvalue λ. Then by definition, $\vec{v} \neq \vec{0}$ and

$$P\vec{v} = \lambda\vec{v} \quad (1).$$

Also:

$$P^2\vec{v} = P(P\vec{v}) = P(\lambda\vec{v}) = \lambda(P\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v} \quad (2).$$

Since it is given that $P^2 = P$, compare (1) and (2) to conclude that

$$\lambda\vec{v} = \lambda^2\vec{v} \Leftrightarrow (\lambda - \lambda^2)\vec{v} = \vec{0}.$$

Since $\vec{v} \neq \vec{0}$, we must have

$$\lambda - \lambda^2 = 0 \Leftrightarrow \lambda(1 - \lambda) = 0 \Leftrightarrow \lambda = 0 \text{ or } \lambda = 1.$$

8. [avg: 6.25/10] Indicate if the following statements are **True** or **False**, and give a *brief* explanation why.

(a) [2 marks] The line with vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ is orthogonal to the plane with scalar equation $x - y + z = 6$. ○ **True** ⊗ **False**

Explanation: the line is actually *parallel* to the plane since $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$.

(b) [2 marks] If A is a 3×3 matrix such that $A^T = -A$, then A is not invertible.

⊗ **True** ○ **False**

Explanation: $\det(A) = \det(A^T) = \det(-A) = (-1)^3 \det(A) = -\det(A) \Rightarrow \det(A) = 0$.

(c) [2 marks] $\det \begin{bmatrix} 1 & 3 & -4 & 6 \\ 2 & 5 & 10 & 4 \\ 1 & 3 & -4 & 6 \\ 7 & 1 & 4 & 0 \end{bmatrix} = 0$. ⊗ **True** ○ **False**

Explanation: the first and third rows are identical, so the determinant is zero.

(d) [2 marks] The vector $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 1 & -4 & 1 \\ 2 & 3 & -1 \\ 2 & 1 & 1 \end{bmatrix}$.

⊗ **True** ○ **False**

Explanation: $A\vec{v} = 2\vec{v}$, so $\vec{v}$ is an eigenvector of A (with eigenvalue $\lambda = 2$.)

(e) [2 marks] If A is an invertible matrix then A is diagonalizable. ○ **True** ⊗ **False**

Counter example: the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible, because $\det(A) = 1 \neq 0$, but it is not diagonalizable. *It's the example in the book of a non-diagonalizable matrix.* Or you can observe that its only eigenvalue is $\lambda = 1$, repeated, but there is only one basic eigevenvector, since the solution to the homogeneous system of equations with augmented matrix $\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ only has one parameter.

Some Alternate Solutions:

In Question 2(b) the determinant could be calculated in many different ways. For example,

- by the formula

$$\det\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = 1+0+0-2-0-2 = -3$$

- or by a cofactor expansion

$$\det\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = -2\det\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + 0 + \det\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = -2+0-1 = -3$$

In Question 3(b), students could use the adjugate formula for the inverse. First you need the determinant of A :

$$\det(A) = \det\begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix} = 14+12+18-12-21-12 = -1.$$

Then the adjugate of A is

$$\mathrm{adj}\begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ -2 & -2 & -3 \\ 0 & -3 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -2 & -3 \\ -1 & -3 & -1 \end{bmatrix},$$

and

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) = -\begin{bmatrix} -1 & -2 & 0 \\ 0 & -2 & -3 \\ -1 & -3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}.$$

In Question 6(a) you could find the intersection point Q of the line $\mathcal{L}$ and the plane $\mathcal{P}$. Then the minimum distance is $D = \|\overrightarrow{QP}\|$. The line $\mathcal{L}$ through P and Q has direction vector $\vec{n}$, so its parametric equations are

$$x = 2 + 2t, y = 3 + t, z = -t.$$

Substitute these into the equation of the plane and solve for t :

$$2(2 + 2t) + (3 + t) - (-t) = 6 \Leftrightarrow 6t = -1 \Leftrightarrow t = -\frac{1}{6}.$$

So Q has coordinates $(2 - 2(-1/6), 3 - 1/6, 1/6)$ and then

$$D = \|\overrightarrow{PQ}\| = \left\| \begin{bmatrix} -2/6 \\ 1/6 \\ -1/6 \end{bmatrix} \right\| = \frac{1}{6}\sqrt{6} = \frac{1}{\sqrt{6}}.$$

In Question 6(b) you could use the good old Pythagorean Theorem to find D :

$$D^2 + \|\mathrm{proj}_{\vec{d}}(\overrightarrow{PQ})\|^2 = \|\overrightarrow{PQ}\|^2.$$

So

$$D^2 = \left\| \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix} \right\|^2 - \left\| \frac{1}{2} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\|^2 = 17 - \frac{7}{2} = \frac{27}{2} \Rightarrow D = \sqrt{\frac{27}{2}} = 3\sqrt{\frac{3}{2}} \text{ or } \frac{3}{2}\sqrt{6}$$

In Question 6(b) the normal vector to the plane that contains both lines is

$$\vec{n} = \overrightarrow{PQ} \times \vec{d} = \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ 6 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}.$$

Then a normal vector that is on a line that connects both lines is

$$\vec{w} = \vec{n} \times \vec{d} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 7 \\ -7 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Then the minimum distance between the two lines is

$$D = \|\mathrm{proj}_{\vec{w}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \vec{w}|}{\|\vec{w}\|} = \frac{9}{\sqrt{6}} = \frac{3\sqrt{6}}{2}.$$

In Question 8(a) you can say that the given line is not orthogonal to the given plane because the direction vector of the line is not parallel to the normal vector of the plane; that is,

$$\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \neq k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

The end.