University of Toronto

SOLUTIONS to MAT188H1F TERM TEST 1

of **Tuesday, October 14, 2008**

Duration: 90 minutes

TOTAL MARKS: 60

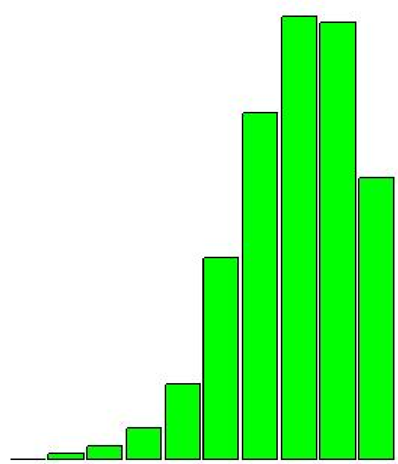
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

- Many, many students lost marks needlessly for incorrect notation. Mathematics has its own set of symbols. If you are going to use them, you must use them correctly. Incorrect notation will cost you marks.
- Questions 1, 2, 3, 4 and 6 are practically identical to the corresponding questions on last year's test.
- Questions 1, 2, 3, 4, 5(a) and 6(a) are all completely routine computations, representing 41 marks that everybody should have aced.
- Only questions 5(b), 6(b) and 7 required some thought. The solution to 6(b) is related to 6(a), but many students missed the connection.
- In Question 7(c) you can't multiply both sides of the given equation by A^{-1} because that assumes A^{-1} exists, which is what you are supposed to establish.

Breakdown of Results: 862 students wrote this test. The marks ranged from 16.6% to 100%, and the average was 73.2%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	15.3%
A	39.1%	80-89%	23.8%
B	24.1%	70-79%	24.1%
C	18.9%	60-69%	18.9 %
D	11.1%	50-59%	11.1 %
F	6.8%	40-49%	4.1%
		30-39%	1.7 %
		20-29%	0.7%
		10-19%	0.3%
		0-9%	0.0%



1. [8 marks] Find the reduced row-echelon form of the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 3 & -1 \\ -3 & 6 & -4 & -9 & 3 \\ -1 & 2 & -2 & -4 & -3 \\ 1 & -2 & 2 & 2 & -5 \end{bmatrix}.$$

What is the rank of A?

Solution:

$$\begin{bmatrix} 1 & -2 & 1 & 3 & -1 \\ -3 & 6 & -4 & -9 & 3 \\ -1 & 2 & -2 & -4 & -3 \\ 1 & -2 & 2 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 3 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & -1 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 1 & 0 & -13 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 & 0 & -13 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of A is the number of leading 1's in the reduced row-echelon matrix; so the rank is 3.

Note: The reduced row-echelon form of a matrix is unique; there is only one correct answer.

2. [8 marks] Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix}$ and use it to solve for X if $AX = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}$.

Solution:

$$
\begin{aligned}
(A|I) = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right]
&\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] = (I|A^{-1})
\end{aligned}
$$

So

$$
A^{-1} = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} \text{ and } X = A^{-1}\begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 10 \\ -2 \end{bmatrix}.
$$

Alternate Method: you could use the adjoint formula to find A^{-1} :

$$
\begin{aligned}
A^{-1} &= \frac{1}{\det A}\,\mathrm{adj}(A) \\
&= \frac{1}{1}\,[C_{ij}(A)]^T \\
&= \begin{bmatrix} 2 & 0 & -1 \\ -(3) & 2 & -(-1) \\ 1 & -(1) & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 2 & -3 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

3. [8 marks] Find the determinant of the matrix

$$A = \begin{bmatrix} 4 & 3 & 0 & 1 \\ -1 & 1 & 1 & 2 \\ 3 & 0 & 2 & -1 \\ -1 & 2 & 2 & 1 \end{bmatrix}.$$

Solution: Use a combination of row or column operations and cofactor expansions. For example:

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 4 & 3 & 0 & 1 \\ -1 & 1 & 1 & 2 \\ 3 & 0 & 2 & -1 \\ -1 & 2 & 2 & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} 7 & 0 & -3 & -5 \\ -1 & 1 & 1 & 2 \\ 3 & 0 & 2 & -1 \\ 1 & 0 & 0 & -3 \end{bmatrix} \\
&= (1)(-1)^{2+2} \det \begin{bmatrix} 7 & -3 & -5 \\ 3 & 2 & -1 \\ 1 & 0 & -3 \end{bmatrix} \\
&= \det \begin{bmatrix} 7 & -3 & 16 \\ 3 & 2 & 8 \\ 1 & 0 & 0 \end{bmatrix} \\
&= (1)(-1)^{3+1} \det \begin{bmatrix} -3 & 16 \\ 2 & 8 \end{bmatrix} \\
&= (-24 - 32) \\
&= -56
\end{aligned}$$

4. [8 marks] Find all the solutions to the system of homogeneous equations

$$\left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & - & x_4 & + & 3x_5 & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & 0 \\ 2x_1 & + & 4x_2 & + & 2x_3 & - & x_4 & + & 7x_5 & = & 0 \end{array}\right.$$

and express your answer as a linear combination of basic solutions.

Solution:

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 1 & 2 & 0 \\ 2 & 4 & 2 & -1 & 7 & 0 \end{array}\right] \to \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Let $x_2 = s$ and $x_5 = t$ be parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 7t \\ s \\ 3t \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

The solution must be expressed as a linear combination to get full marks.

5. [9 marks] The parts of this question are unrelated.

(a) [5 marks] Use Cramer's Rule to solve for x_2 in the system of equations

$$\left\{ \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 4 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 5 \\ -x_1 & + & x_2 & - & x_3 & = & 1 \end{array} \right.$$

Solution: Must use Cramer's Rule; any other method is worthless.

$$\begin{aligned} x_2 &= \frac{\det \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 3 \\ -1 & 1 & -1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ -1 & 1 & -1 \end{bmatrix}} \\ &= \frac{\det \begin{bmatrix} 0 & 5 & 0 \\ 0 & 7 & 1 \\ -1 & 1 & -1 \end{bmatrix}}{\det \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}} = \frac{-5}{-2} = \frac{5}{2} \end{aligned}$$

(b) [4 marks] Suppose A is a 3×3 matrix such that $\det A = -2$. What is the value of $\det(\operatorname{adj} A)$?

Solution: Use the adjoint formula.

$$\begin{aligned} A \operatorname{adj} A = \det(A)\, I \quad &\Rightarrow \quad \det\left(A \operatorname{adj} A\right) = \det(-2I) \\ &\Rightarrow \quad \det A \, \det(\operatorname{adj} A) = (-2)^3 = -8 \\ &\Rightarrow \quad \det(\operatorname{adj} A) = \frac{-8}{\det A} = \frac{-8}{-2} \\ &\Rightarrow \quad \det(\operatorname{adj} A) = 4 \end{aligned}$$

6. [10 marks] Consider the system of equations with unknowns x, y and z :

$$\begin{cases} x & + & y & + & z & = & 1 \\ x & + & (a+2)y & + & 2z & = & 5 \\ x & + & 7y & + & (a+1)z & = & 9 \end{cases}$$

(a) [4 marks] Find the determinant of the coefficient matrix of the above system.

Solution:

$$\begin{aligned} \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & a+2 & 2 \\ 1 & 7 & a+1 \end{bmatrix} &= \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & a+1 & 1 \\ 0 & 6 & a \end{bmatrix} \\ &= a(a+1) - 6 \\ &= a^2 + a - 6 \\ &= (a+3)(a-2) \end{aligned}$$

(b) [6 marks] For which values of a does the above system have *(i)* no solution? *(ii)* a unique solution? *(iii)* infinitely many solutions?

Solution:

(ii) The system has a unique solution if the coefficient matrix is invertible

$$\Leftrightarrow (a+3)(a-2) \neq 0 \Leftrightarrow a \neq -3 \text{ or } a \neq 2.$$

(i) If $a = -3$, reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 5 \\ 1 & 7 & -2 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -4 \\ 0 & 6 & -3 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

So in this case there are no solutions.

(iii) If $a = 2$, reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 4 & 2 & 5 \\ 1 & 7 & 3 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 6 & 2 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this case one parameter is needed, so there are infinitely many solutions.

7. [9 marks; 3 mark for each part] Determine if the following statements are True or False. Circle your choice. Justify your choice with a *brief* explanation.

(a) If one solution to a system of linear equations is the trivial solution, then the system is homogeneous. **True** **False**

Solution: True. Let the system be $AX = B$. If $X = O$ is a solution then

$$B = AO = O,$$

and the system is homogeneous.

(b) If A is a square matrix such that $A^2 = A$, then $A = I$ or $A = O$. **True** **False**

Solution: False. Find a counterexample. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

then $A \neq I, A \neq O$, but

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A.$$

(c) If A is a square matrix such that $A^2 - A - 2I = O$, then A is invertible and $A^{-1} = \dfrac{1}{2}(A - I)$. **True** **False**

Solution: True.

$$\begin{aligned} A^2 - A - 2I = O &\Rightarrow A^2 - A = 2I \\ &\Rightarrow A(A - I) = 2I \\ &\Rightarrow A\left(\frac{1}{2}(A - I)\right) = I \\ &\Rightarrow A^{-1} = \frac{1}{2}(A - I), \text{ by definition of inverse.} \end{aligned}$$