University of Toronto
Solutions to **MAT188H1F TERM TEST**
of **Tuesday, October 29, 2013**
Duration: 100 minutes

Only aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments about the Test:

1. In Question 1, you should follow instructions: find A^{-1} and use it to solve the equations in parts (b) and (c). If you solved the equations in (b) and (c) without using A^{-1} you could get at most one mark for each part, and that only if your answer for $\mathbf{x}$ is correct!

2. Unfortunately in Question 2 there was a typographical error in A : the entry in the second row, fifth column should have been $+3$, not -3. However, there is no need to check R; it was given so that you wouldn't have to do any row reduction at all. See solutions. Nevertheless, some of you actually did the reduction

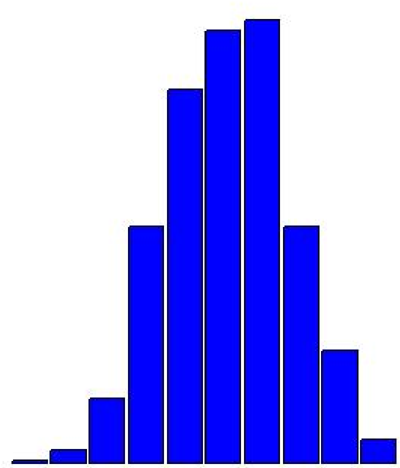
$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & -3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 9 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = R.$$

   Answers for parts (a), (b) and (c) will be considered correct based on *either* version of R, even if your answers from one part contradict the answer from another part.

3. Its not necessary to use the definition of a subspace to do Question 5(a). In general, it is usually more convenient to show a set S is a subspace by showing S is the null space of some matrix, or that S is the span of some vectors.

Breakdown of Results: 893 students wrote this test. The marks ranged from 8.3% to 96.7%, and the average was 55.4%, surprisingly low. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	1.2%
A	7.0%	80-89%	5.8%
B	12.2%	70-79%	12.2%
C	22.8%	60-69%	22.8%
D	22.3%	50-59%	22.3%
F	35.7%	40-49%	19.3%
		30-39%	12.2%
		20-29%	3.4%
		10-19%	0.7%
		0-9%	0.1%



1. [10 marks] Let $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -5 & 4 \\ -1 & 3 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$.

(a) [4 marks] Find the inverse of A

Solution: use the Gaussian algorithm.

$$(A|I) = \left[\begin{array}{ccc|ccc} 1 & -3 & 1 & 1 & 0 & 0 \\ 2 & -5 & 4 & 0 & 1 & 0 \\ -1 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & 7 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right] = (I|A^{-1})$$

So

$$A^{-1} = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

and use it to solve the following equations for $\mathbf{x}$:

(b) [2 marks] $A\mathbf{x} = \mathbf{b}$

Solution:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \\ 1 \end{bmatrix}.$$

(c) [4 marks] $A^T A\mathbf{x} = \mathbf{b}$

Solution:

$$\mathbf{x} = (A^T A)^{-1}\mathbf{b} = A^{-1}(A^T)^{-1}\mathbf{b} = A^{-1}(A^{-1})^T\mathbf{b}$$

$$= \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 0 \\ 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 80 \\ 21 \\ -13 \end{bmatrix}.$$

2. [10 marks] Given that the reduced echelon form of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -2 & 0 & 9 & -16 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ is defined by $T(\mathbf{x}) = A\mathbf{x}$, find the following:

(a) [4 marks] a basis for null(A)

Solution: null$(A) =$ null(R). Let $x_2 = s, x_4 = t, x_5 = u$ be parameters. Then

$$\mathbf{x} \in \text{null}(A) \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - 9t + 16u \\ s \\ 3t - 5u \\ t \\ u \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

A basis of null(A) is the set of independent vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) [3 marks] a basis for range(T)

Solution: range(T) is spanned by the columns of A. From R columns 2, 4 and 5 of A are linear combinations of columns 1 and 3 of A. Therefore, a basis for range(T) is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$$

(c) [3 marks] the general solution $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{b}$, for $\mathbf{b} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$.

Solution: the general solution is any particular solution plus the solution to the corresponding homogeneous system. Since $\mathbf{b}$ is the second column of A, an obvious particular solution is $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0$. Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

Another obvious particular solution is $x_1 = -2, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0$ since $\mathbf{b}$ is negative two times the first column of A.

3. [10 marks] Let $S : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ and $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be defined by the formulas

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 - 2x_3 \\ x_1 + x_3 \\ x_1 - x_2 \end{bmatrix}, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_4 \end{bmatrix}.$$

(a) [3 marks] Is S one-to-one?

Solution: Yes. Let the matrix of S be A. Then kernel(S) = null(A) and

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \\ 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So null$(A) = \{\mathbf{0}\}$ and S must be one-to-one.

(b) [3 marks] Is T onto?

Solution: Yes. Let the matrix of T be B. Then range(T) is spanned by the columns of B. We have

$$B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

So the first three columns of B are independent. Three independent vectors in $\mathbb{R}^3$ form a basis of $\mathbb{R}^3$. Hence range$(T) = \mathbb{R}^3$ and T is onto.

(c) [4 marks] Find the formula for $T(S(\mathbf{x}))$ for $\mathbf{x}$ in $\mathbb{R}^3$.

Solution: the matrix of $T \circ S$ is BA. So

$$T(S(\mathbf{x})) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ -x_1 + x_2 - 3x_3 \\ x_2 + x_3 \end{bmatrix}$$

Note: this whole question can be done without matrices, simply by calculating kernel(S), range(T) and $T(S(\mathbf{x}))$ directly.

4. [10 marks] The parts of this question are unrelated.

(a) [5 marks] Show that if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set of vectors in $\mathbb{R}^4$, then so is $\{\mathbf{u}_1, c\,\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_3\}$ for any scalar c.

Solution: suppose $a_1\,\mathbf{u}_1 + a_2(c\,\mathbf{u}_1 + \mathbf{u}_2) + a_3\,\mathbf{u}_3 = \mathbf{0}$, for scalars a_i. Then

$$(a_1 + ca_2)\mathbf{u}_1 + a_2\,\mathbf{u}_2 + a_3\,\mathbf{u}_3 = \mathbf{0}.$$

Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly independent set it follows that

$$a_1 + ca_2 = 0, a_2 = 0, a_3 = 0$$

$$\Rightarrow a_1 = 0, a_2 = 0, a_3 = 0.$$

Hence $\{\mathbf{u}_1, c\,\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_3\}$ is also a linearly independent set.

(b) [5 marks] Set up the following problem

> A total of 385 people attend the premiere of a new movie. Ticket prices are \$11 for adults and \$8 for children. If the total revenue is \$3974, how many adults and children attended?

as a system of linear equations, and solve it.

Solution: let x be the number of adults who attended; let y be the number of children who attended. Then the revenue is $11x + 8y$. We have

$$\begin{cases} x + y = 385 \\ 11x + 8y = 3974 \end{cases}$$

Solve this any way you like. For instance:

$$\left[\begin{array}{cc|c} 1 & 1 & 385 \\ 11 & 8 & 3974 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 385 \\ 0 & 3 & 261 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 385 \\ 0 & 1 & 87 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 298 \\ 0 & 1 & 87 \end{array}\right]$$

So 298 adults and 87 children attended the premiere.

5. [10 marks] Let S be the set of vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ in $\mathbb{R}^4$ such that $x_1 - x_2 + x_3 - x_4 = 0$.

(a) [6 marks] Show that S is a subspace of $\mathbb{R}^4$ and find its dimension.

Solution: $S = \text{null}\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$, so S is a subspace of $\mathbb{R}^4$. To find a basis of S solve the system $x_1 - x_2 + x_3 - x_4 = 0$: let $x_2 = s, x_3 = t, x_4 = u$ be parameters, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - t + u \\ s \\ t \\ u \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and

$$S = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\},$$

which is another way to show that S is a subspace. In particular the three spanning vectors are independent, and so form a basis for S. Thus $\dim(S) = 3$.

(b) [4 marks] Find a linear transformation $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that range$(T) = S$ and

$$\text{kernel}(T) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Solution: let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be the standard basis of $\mathbb{R}^4$. Then the matrix of T is

$$A = \begin{bmatrix} T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid T(\mathbf{e}_3) \mid T(\mathbf{e}_4) \end{bmatrix}.$$

Since kernel$(T) = \text{span}\{\mathbf{e}_1\}$, we must have $T(\mathbf{e}_1) = \mathbf{0}$. For the remaining three columns of A let $T(\mathbf{e}_2), T(\mathbf{e}_3), T(\mathbf{e}_4)$ be the three basis vectors of S, in any order. For instance, you could take

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $T(\mathbf{x}) = A\,\mathbf{x}$ and range$(T) = \text{span}\{T(\mathbf{e}_2), T(\mathbf{e}_3), T(\mathbf{e}_4)\} = S$.

6. [10 marks; 2 marks each.] Indicate if the following statements are True or False, and give a brief explanation why. Circle your choice.

(a) If A and B are invertible $n \times n$ matrices, then $A + B$ is also invertible.

True **False**

Solution: False. If $A = I, B = -I$, then both A and B are invertible, but $A + B$ is the zero matrix which is not invertible.

(b) If A is an $n \times n$ matrix such that $A^3 - 2A - 4I = 0$, where I and 0 are the $n \times n$ identity and zero matrices, respectively, then A is invertible.

True **False**

Solution: True.

$$A^3 - 2A - 4I = 0 \Rightarrow A(A^2 - 2I) = 4I \Rightarrow A\left(\frac{1}{4}(A^2 - 2I)\right) = I,$$

which means A is invertible (and its inverse is $\frac{1}{4}(A^2 - 2I)$.)

(c) $\dim\left(\operatorname{span}\left\{\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}\right\}\right) = 3$ **True** **False**

Solution: False. If it were true the three given vectors would be a basis for $\mathbb{R}^3$, and so would be independent. But they are dependent:

$$\begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

(d) $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \\ 4 \end{bmatrix}\right\} = \mathbb{R}^4$ **True** **False**

Solution: False. It takes at least four vectors to span $\mathbb{R}^4$.

(e) If A is an $n \times n$ matrix, then the set S consisting of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = 3\mathbf{x}$, is a subspace of $\mathbb{R}^n$. **True** **False**

Solution: True. $S = \operatorname{null}(A - 3I)$.