University of Toronto
Faculty of Arts and Science

MAT221H1F
Applied Linear Algebra

Final Examination
December 2013

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Duration: 3 hours

Last Name: ____________________

Given Name: ____________________

Student Number: ____________________

No calculators or other aids are allowed.

FOR MARKER USE ONLY	
Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
TOTAL	/80

1. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 2x_2 - x_3 - 4x_4 \\ x_1 + x_2 - 2x_3 - 3x_4 \\ 2x_1 - 3x_2 + x_3 - x_4 \end{bmatrix}.$$

(a) Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$.

(b) Find a basis for the range of T. Is T onto?

2. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \mid x_1 + x_3 = x_4 \right\}$.

(a) Show that W is a subspace of $\mathbb{R}^4$.

(b) Find a basis for W and determine $\dim(W)$.

3. Find a basis for the rowspace, columnspace, and nullspace of $A = \begin{bmatrix} 0 & 2 & 3 \\ -1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

4(a) Evaluate $\det \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 & 6 \end{bmatrix}$.

4(b) Find all values of c such that the matrix $\begin{bmatrix} c & 2 & 1 \\ 0 & 3 & c \\ 2 & -4 & 1 \end{bmatrix}$ is invertible.

5. Let $W = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find an orthonormal basis for W.

(b) Find a basis for $W^{\perp}$.

(c) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\operatorname{proj}_W(\mathbf{x})$.

6. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Find the eigenvalues of A and a basis for each of the corresponding eigenspaces.

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

7. Consider the quadratic form $q(\mathbf{x}) = 11x^2 + 4xy + 14y^2$.

(a) Express q as $\mathbf{x}^T A\mathbf{x}$ where A is a symmetric matrix, and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

(b) Find new variables $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ that will diagonalize q and find $q(\mathbf{x}')$.

(c) Using your answer from part (b), sketch the graph of $11x^2 + 4xy + 14y^2 = 60$ clearly indicating the new $x' - y'$ axes relative to the original $x - y$ axes.

8. Find the equation $y = a + bx$ of the least-squares line that best fits the points $(-1, 0), (0, 1), (1, 2)$, and $(2, 4)$.