Wednesday, December 17 | **START: 9:00 AM** | **DURATION: 180 mins**

University of Toronto

Department of Mathematics

FINAL EXAMINATION, DECEMBER 2014
MAT221H1F
Applied Linear Algbera

Solutions

EXAMINERS: D. Burbulla, S. Uppal

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Student NUMBER: ________________________________

Student SIGNATURE: ________________________________

Instructions.

1. There are **100** possible marks to be earned in this exam. The examination booklet contains a total of 13 pages. It is your responsibility to ensure that *no pages are missing from your examination.* Do not detach any pages from your examination.
2. You may write in black ink, blue ink, or in pencil. Do not write in red ink, and ensure that your solutions are LEGIBLE.
3. No aids of any kind are permitted. CALCULATORS AND OTHER ELECTRONIC DEVICES (INCLUDING PHONES) ARE NOT PERMITTED.
4. Have your student card ready for inspection.
5. There are no part marks for Multiple Choice (MC) questions.
6. For the full answer questions, write the solutions on the question pages themselves. You may use the backs of pages for rough work. The backs of pages WILL NOT BE MARKED unless you *clearly* indicate otherwise on the question pages.
7. For the full answer questions, show all of your work *but do not include extraneous information.*

FOR MARKER'S USE ONLY:

MC	Q11	Q12	Q13	Q14	Q15	Q16	Q17	Q18	**TOTAL**

Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice.

Each question is worth 2 marks. For each question, choose the BEST option from the given options.

1. If S is a subspace of $\mathbb{R}^7$ and $\dim(S) = 4$, then $\dim(S^{\perp}) =$

(A) 2

$\boxed{\textbf{(B) } 3}$

(C) 4

(D) 5

(E) 6

2. In the solution to the system of equations $\begin{cases} 3x + 4y - 3z = 6 \\ x + 3y + z = -2 \\ -x - 2y + 4z = -8 \end{cases}$ the value of y is

$\boxed{\textbf{(A) } 0}$

(B) $\frac{6}{19}$

(C) $-\frac{6}{19}$

(D) 2

(E) not uniquely determined.

3. Which of the following is **not** equivalent to the statement: "A is an invertible $n \times n$ matrix."

(A) $\det(A) \neq 0$.

(B) $\lambda = 0$ is not an eigenvalue of A.

(C) $\text{col}(A) = \mathbb{R}^n$

$\boxed{\textbf{(D) } A \text{ is diagonalizable.}}$

(E) The rank of A is n.

4. Let S be the set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $xyz = 0$. Which of the following statements is true?

(A) S is the zero subspace.

(B) S is a non-trivial subspace of $\mathbb{R}^3$.

(C) S does not contain the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{\textbf{(D) } S \text{ is closed under scalar multiplication.}}$

(E) S is closed under vector addition.

5.

$$\dim\left(\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ -1 \end{bmatrix}\right\}\right) =$$

(A) 1

(B) 2

$\boxed{\textbf{(C) } 3}$

(D) 4

(E) 5

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 5y \\ 4x + 6y \end{bmatrix}.$$

Then the area of the image of the unit square under T is

(A) -8

$\boxed{\textbf{(B)}\ 8}$

(C) -32

(D) 32

(E) 0

7. Let

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

The characteristic polynomial of A is:

$\boxed{\textbf{(A)}\ x^3 - 11x^2 + 38x - 40}$

(B) $x^3 + 11x^2 + 38x + 40$

(C) $x^2 - 11x + 40$

(D) $x^2 + 11x - 40$

(E) $x^3 - 11x^2 - 38x - 40$

8. If A and B are 3×3 matrices such that $\det(A) = 1$ and $\det(B) = -3$, then $\det(-2A^2B^{-1}A^TB^3) =$

$\boxed{\textbf{(A)}\ -72}$

(B) 72

(C) -18

(D) 18

(E) -36

9. Let $\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Then $[\mathbf{x}]_{\mathcal{B}}$, the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$, is

(A) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(B) $\begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

(D) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

(E) $\boxed{\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}}$

10. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ -3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Then $\text{proj}_{\mathbf{v}}\mathbf{u} =$

(A) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(B) $5 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(C) $\frac{5}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(D) $\boxed{\frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}$

(E) $\frac{5}{17} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

Part II - Written Answer Questions. Write your solutions in the space provided for each question.

Each question is worth 10 marks. If a question has more than one part, the value of each part is indicated in brackets.

11. Given that the reduced echelon form of

$$A = \begin{bmatrix} 1 & -2 & -1 & 1 & -2 \\ 4 & -8 & -4 & 5 & -7 \\ 2 & -4 & -2 & 3 & -3 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -2 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. (Put your answers in the blanks to the right.)

(a) [1 mark] dim(row(A)) $\underline{\quad 2 \quad}$

(b) [1 mark] dim(col(A)) $\underline{\quad 2 \quad}$

(c) [1 mark] dim(null(A)) $\underline{\quad 3 \quad}$

(d) [2 marks] A basis for the row space of A $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(e) [2 marks] A basis for the column space of A $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$

(f) [3 marks] A basis for the null space of A $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

12. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ and $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be linear transformations defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ 3x_1 + 2x_2 \\ -x_1 + x_2 \end{bmatrix}, \quad S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 - x_4 \\ -x_1 + x_2 + x_3 + 2x_4 \end{bmatrix}.$$

Put your answers to the following questions in the blanks to the right.

(a) [1 mark] What is the matrix of T? $\begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}$

(b) [1 mark] What is the matrix of S? $\begin{bmatrix} 1 & -2 & 1 & -1 \\ -1 & 1 & 1 & 2 \end{bmatrix}$

(c) [1 mark] Is T one-to-one? Yes

(d) [1 mark] Is T onto? No

(e) [1 mark] Is S one-to-one? No

(f) [1 mark] Is S onto? Yes

(g) [2 marks] Is $S \circ T$ one-to-one, where $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$? Yes

The matrix of $S \circ T$ is $\begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix}$, which is invertible.

(h) [2 marks] Is $T \circ S$ onto, where $(T \circ S)(\mathbf{x}) = T(S(\mathbf{x}))$? No

The matrix of $T \circ S$ is $\begin{bmatrix} 3 & -5 & 1 & -4 \\ 0 & -1 & 2 & 1 \\ 1 & -4 & 5 & 1 \\ -2 & 3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 1 & -4 \\ 0 & -1 & 2 & 1 \\ 0 & 7 & -14 & -7 \\ 0 & -5 & 10 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & 1 & -4 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

13. The parts of this question are unrelated.

(a) [4 marks] Find a basis for $S^\perp$ if $S = \operatorname{span}\left\{\begin{bmatrix} 1 & 0 & -1 & 2 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 5 & 3 \end{bmatrix}^T\right\}$.

Solution:

$$S^\perp = \operatorname{null}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 5 & 3 \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Since the dimension of $S^\perp$ is $4-2=2$, a basis of S is

$$\left\{\begin{bmatrix} 1 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

(b) [3 marks] Find all values of a for which $A = \begin{bmatrix} 1 & a & 0 \\ 0 & 2 & 2 \\ a & 12 & 3 \end{bmatrix}$ is not invertible.

Solution:

$$0 = \det\begin{bmatrix} 1 & a & 0 \\ 0 & 2 & 2 \\ a & 12 & 3 \end{bmatrix} = 6 + 2a^2 - 24 = 2a^2 - 18 \Rightarrow a^2 = 9 \Rightarrow a = \pm 3.$$

(c) [3 marks] Let $T : \mathbb{R}^7 \longrightarrow \mathbb{R}^5$ be a linear transformation. Explain why T cannot be one-to-one.

Solution: use the rank-nullity theorem. Let the matrix of T be A; then A is a 5×7 matrix. We have

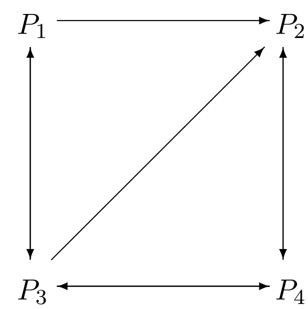
$$\dim(\operatorname{null}(A)) + \dim(\operatorname{row}(A)) = 7.$$

If T is one-to-one, then $\dim(\operatorname{null}(A)) = 0$; consequently $\dim(\operatorname{row}(A)) = 7$. Since A only has 5 rows, it is impossible for it to have 7 independent rows. So T cannot be one-to-one.

OR: do it directly. The rank of A is at most 5 so the nullity of A is at least 2. Thus it is impossible for the nullity of A to be 0.

14.

Consider a simple internet with only four web pages, P_1, P_2, P_3, P_4, linked as in the diagram to the right:



(a) [4 marks] Find the transition matrix $A = [a_{ij}]$, where a_{ij} is the probability that a person at page P_j randomly picks an available link to page P_i.

Solution: The transition matrix is given by

$$A = \begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & 0 \end{bmatrix}$$

(b) [6 marks] Find the equilibrium distribution of users.

Solution: let the equilibrium solution be $\mathbf{x}$, where $\mathbf{x}$ is a probability vector and $A\mathbf{x} = \mathbf{x}$. We have:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ -1/2 & 1 & -1/3 & -1/2 & 0 \\ -1/2 & 0 & 1 & -1/2 & 0 \\ 0 & -1 & -1/3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 5/3 & -1 & 0 \\ 0 & -1 & -1/3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & -1 & -1/3 & -1 & 0 \\ 0 & 0 & 5/3 & -1 & 0 \\ 0 & 0 & -5/3 & 1 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 0 & 0 \\ 0 & -1 & -1/3 & -1 & 0 \\ 0 & 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/5 & 0 \\ 0 & 1 & 0 & -4/5 & 0 \\ 0 & 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right];$$

for which the general solution is

$$\mathbf{x} = \frac{t}{5} \begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix}.$$

This is a probability vector if $t = 5/13$; then

$$\mathbf{x} = \frac{1}{13} \begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix}.$$

So the equilibrium distribution is:

$1/13$-th of users at P_1, $4/13$-th of users at P_2, $3/13$-th of users at P_3, and $5/13$-th of users at P_4.

15. Let $A = \begin{bmatrix} 1/3 & 3/4 \\ 2/3 & 1/4 \end{bmatrix}$.

(a) [6 marks] Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Solution:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1/3 & -3/4 \\ -2/3 & \lambda - 1/4 \end{bmatrix} = \lambda^2 - \frac{7}{12}\lambda - \frac{5}{12} = (\lambda - 1)(\lambda + 5/12).$$

The eigenvalues of A are

$$\lambda_1 = 1 \text{ and } \lambda_2 = -\frac{5}{12}.$$

Find the eigenvectors:

$$\text{null}(\lambda_1 I - A) = \text{null} \begin{bmatrix} 2/3 & -3/4 \\ -2/3 & 3/4 \end{bmatrix} = \text{null} \begin{bmatrix} 8 & -9 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 9 \\ 8 \end{bmatrix} \right\};$$

$$\text{null}(\lambda_2 I - A) = \text{null} \begin{bmatrix} -3/4 & -3/4 \\ -2/3 & -2/3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Take

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -5/12 \end{bmatrix} \text{ and } P = \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix}.$$

(b) [3 marks] Find and simplify a formula for A^n.

Solution:

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -5/12 \end{bmatrix}^n \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-5/12)^n \end{bmatrix} \frac{1}{17} \begin{bmatrix} 1 & 1 \\ -8 & 9 \end{bmatrix} \\ &= \frac{1}{17} \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -8(-5/12)^n & 9(-5/12)^n \end{bmatrix} \\ &= \frac{1}{17} \begin{bmatrix} 9 + 8(5/12)^n & 9 - 9(-5/12)^n \\ 8 - 8(-5/12)^n & 8 + 9(-5/12)^n \end{bmatrix} \end{aligned}$$

(c) [1 marks] What can you say about the entries of A^n as $n \to \infty$?

Solution: as $n \to \infty$,

$$A^n \to \frac{1}{17} \begin{bmatrix} 9 & 9 \\ 8 & 8 \end{bmatrix}.$$

16. Let $S = \operatorname{span}\left\{[\begin{array}{cccc} 1 & 0 & 0 & 1 \end{array}]^T, [\begin{array}{cccc} 0 & 1 & 0 & 1 \end{array}]^T, [\begin{array}{cccc} 1 & 1 & 1 & 1 \end{array}]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

Solution: call the three given vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and apply the Gram-Schmidt algorithm to find an orthogonal basis $\mathbf{f}_2, \mathbf{f}_2, \mathbf{f}_3$. Take $\mathbf{f}_1 = \mathbf{x}_1$,

$$\mathbf{f}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{f}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

Optional: clear fractions and take

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{f}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{f}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

Either way $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\right]^T$. Find $\operatorname{proj}_S(\mathbf{x})$.

Solution: using $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ with fractions cleared.

$$\operatorname{proj}_S \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 = \frac{5}{2}\mathbf{f}_1 + \frac{7}{6}\mathbf{f}_2 + \frac{8}{12}\mathbf{f}_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

Cross-check/Alternate Solution: $S^\perp = \operatorname{span}\{\mathbf{y}\}$ with $\mathbf{y} = [\begin{array}{cccc} 1 & 1 & -1 & -1 \end{array}]^T$. Then

$$\operatorname{proj}_S \mathbf{x} = \mathbf{x} - \operatorname{proj}_{S^\perp}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} = \mathbf{x} + \frac{4}{4}\mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

17. Let S be the set of vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $\det \begin{bmatrix} 1 & 2 & 1 & x_1 \\ 3 & 4 & 5 & x_2 \\ 1 & 1 & 0 & x_3 \\ 0 & -1 & 2 & x_4 \end{bmatrix} = 0$. Show that S is a subspace of $\mathbb{R}^4$ and find its dimension.

Solution: there are lots of ways to do this. Here's one of the shortest. Use the cofactor expansion along column 4:

$$\det \begin{bmatrix} 1 & 2 & 1 & x_1 \\ 3 & 4 & 5 & x_2 \\ 1 & 1 & 0 & x_3 \\ 0 & -1 & 2 & x_4 \end{bmatrix} = x_1 C_{14} + x_2 C_{24} + x_3 C_{34} + x_4 C_{14};$$

thus

$$S = \text{null} \begin{bmatrix} C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix},$$

which automatically means S is a subspace of $\mathbb{R}^4$.

Then

$$\dim(S) = \dim\left(\text{null} \begin{bmatrix} C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix}\right) = 4 - \dim\left(\text{row} \begin{bmatrix} C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix}\right) = 4 - 1 = 3.$$

18. Find the least squares approximating line for the data points $(-2,0),(0,1),(1,1),(1,2),(2,3)$.

Solution: let the line be $y = a + bx$. Let

$$M = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} a \\ b \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Then

$$\begin{aligned} M^T M\,\mathbf{z} = M^T\,\mathbf{y} \quad &\Leftrightarrow \quad \begin{bmatrix} 5 & 2 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \\ &\Leftrightarrow \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 10 \end{bmatrix}^{-1}\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{46}\begin{bmatrix} 52 \\ 31 \end{bmatrix}. \end{aligned}$$

So the best approximating line to the data is

$$y = \frac{26}{23} + \frac{31}{46}x.$$

For interest the points and the line are shown below: