University of Toronto

Solutions to **MAT301H1F TERM TEST, Part 1**

Wednesday, October 26, 2016

Duration: 50 minutes

No aids permitted.

Instructions: Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parentheses beside the question number. **Total Marks: 25**

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1. [5 marks; 1 mark for each part.] Let G be a group with operation 'multiplication' and identity element e. Define the following:

   (a) the center of G

   Solution: the center of G is $\{g \in G \mid gx = xg, \text{ for all } x \in G\}$

   (b) the order of $a \in G$

   Solution: the order of a is the least positive integer n such that $a^n = e$.

   (c) a homomorphism from G to another group H

   Solution: the function $f : G \longrightarrow H$ is a homomorphism if for all $x, y \in G$,

$$f(xy) = f(x)f(y).$$

   (d) the centralizer of $a \in G$

   Solution: the centralizer of $a \in G$ is $\{x \in G \mid xa = ax\}$

   (e) a subgroup of G

   Solution: H is a subgroup of G if H is a non-empty subset of G that is itself a group using the operation of G.

2. [10 marks; 2 marks for each part.] Find the order of the following elements in the following groups. Put your answer in the appropriate blank to the right.

(a) 7 in $\mathbb{Z}_{35}$ **Ans:** 5

Solution: $2 \cdot 7 \neq 0 \bmod 35,\ 3 \cdot 7 \neq 0 \bmod 35,\ 4 \cdot 7 \neq 0 \bmod 35,\ 5 \cdot 7 \equiv 0 \bmod 35$

(b) 5 in $U(16)$ **Ans:** 4

Solution: $|U(16)| = \phi(16) = 16 - 8 = 8$, so the order of 5 must be 1, 2, 4, or 8.

$$5^2 = 25 \neq 1 \bmod 16,\ 5^4 = 625 \equiv 1 \bmod 16$$

(c) $\begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}$ in D_{16} **Ans:** 8

Solution: this matrix represents a rotation of $\pi/4$ about the origin, so its order is

$$\frac{2\pi}{\pi/4} = 8.$$

Or, say it represents a rotation of $45°$, so its order is $360/45 = 8$.

(d) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 3 & 6 & 7 & 5 \end{pmatrix}$ in S_7 **Ans:** 12

Solution: write the permutation in cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 3 & 6 & 7 & 5 \end{pmatrix} = (1243)\,(567).$$

So its order is $\text{lcm}(4,3) = 12$.

(e) f in $\text{Aut}\,(Z_8)$, defined by $f(1) = 5$. **Ans:** 2

Solution: $\mathbb{Z}_8 = \langle 1 \rangle$ and f is determined by what it does to a generator.

$$f(f(1)) = f(5) = 5 \cdot f(1) = 5 \cdot 5 = 25 \equiv 1 \bmod 8,$$

so the order of f is 2. Or: express f as a permutation of the numbers $0, 1, \ldots, 7$:

$$0 \to 0, 1 \to 5, 2 \to 10 \equiv 2 \bmod 8, 3 \to 15 \equiv 7 \bmod 8,$$

$$4 \to 20 \equiv 4 \bmod 8, 5 \to 25 \equiv 1 \bmod 8, 6 \to 30 \equiv 6 \bmod 8, 7 \to 35 \equiv 3 \bmod 8.$$

That is, f can be considered as the permutation $(0)(15)(2)(37)(4)(6)$, which has order 2.

3. [10 marks; 2 marks for each part.] Determine if the following statements are True or False and give a brief explanation why. Circle your choice to the right.

(a) If H and K are both subgroups of a group G, such that $|H| = 7$ and $|K| = 13$, then $|H \cap K| = 1$. **[True]** **False**

Solution: by Lagrange's Theorem, $|H \cap K|$ divides both $|H| = 7$ and $|K| = 13$, which are relatively prime, so $|H \cap K| = 1$.

(b) $U(16)$ is cyclic. **True** **[False]**

Solution: $U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\}$. If $U(16)$ were cyclic, it would only have $\phi(2) = 1$ element of order 2. But both 7 and 9 have order 2:

$$7^2 = 49 \equiv 1 \bmod 16, \ 9^2 = 81 \equiv 1 \bmod 16.$$

(c) $|U(36)| = 12$ **[True]** **False**

Solution: $|U(36)| = \phi(36) = \phi(4)\phi(9) = (4 - 2)(9 - 3) = 12$.

(d) $\mathbb{Z}_8 \approx D_4$ **True** **[False]**

Solution: both groups have order 8, but $\mathbb{Z}_8$ is Abelian and D_4 isn't.

(e) Every group of order 60 must have an element of order 15. **True** **[False]**

Solution: consider A_5, the alternating group of degree 5. Its order is $5!/2 = 60$. But it has no element of order 15. Consider the possible cycle structure of elements in A_5 : they must be even. The possibilities are

- ϵ, which has order 1
- two 2-cycles, $(ab)(cd)$, which has order 2
- a 3-cycle, (abc), which has order 3
- a 5-cycle, $(abcde)$, which has order 5

So A_5 has no element of order 15.