UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2012 EXAMINATIONS

MAT335H1F Solutions
Chaos, Fractals and Dynamics
Examiner: D. Burbulla

Duration - 3 hours
Examination Aids: A Scientific Hand Calculator

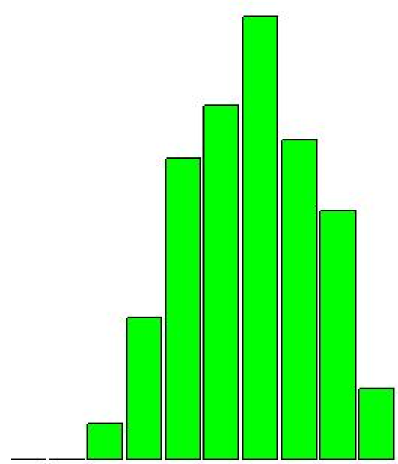
INSTRUCTIONS: All six questions have equal weight. Attempt only five of them. Present your solutions in the exam booklets provided. The marks for each question are indicated in parentheses beside the question number. **MARKS:** 100

General Comments:

1. The least popular questions were numbers 1 and 4.
2. Very few people realized the connection between the Mandelbrot set and parts (c), (d) and (e) of question 5.
3. Questions 1, 3, 4 and 6 were basically taken right out of the book and/or homework.

Breakdown of Results: 108 students wrote this exam. The marks ranged from 25% to 95%, and the average was 62.0%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.7%
A	16.7%	80-89%	13.0%
B	16.7%	70-79%	16.7%
C	23.1%	60-69%	23.1%
D	18.5%	50-59%	18.5%
F	25.0%	40-49%	15.7%
		30-39%	7.4%
		20-29%	1.9%
		10-19%	0.0%
		0-9%	0.0%



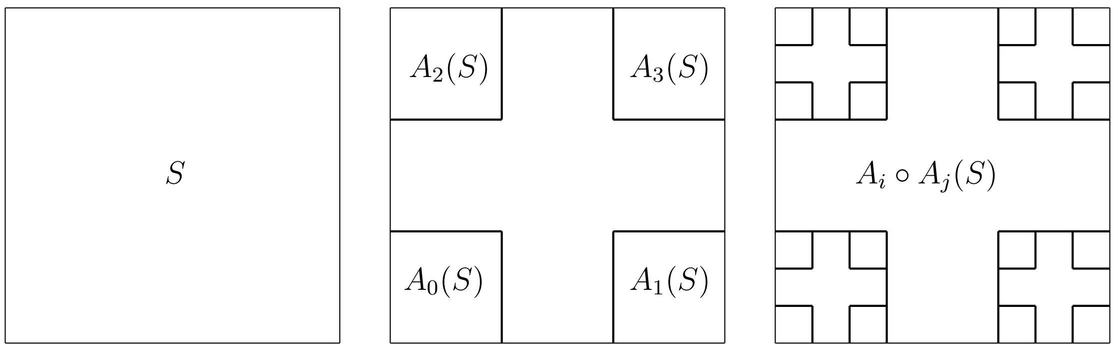
1. [20 marks] Let A_i for $i = 0, 1, 2, 3$ be linear contractions with contraction factor $\beta = 1/3$ and fixed points

$$p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, p_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, p_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

respectively. Let $\mathcal{A}$ be the attractor generated by the iterated function system A_0, A_1, A_2, A_3.

(a) [10 marks] Show that $\mathcal{A} = K \times K$, where K is the Cantor middle-thirds set.

Solution: Let S be the square with vertices p_0, p_1, p_2, p_3 and consider the image of S under $A_i, A_i \circ A_j, \ldots$



Since we are removing middle thirds along the x and y-axes, the attractor is $K \times K$, where K is the Cantor set.

(b) [5 marks] What is the fractal dimension of $\mathcal{A}$?

Solution: at each step we obtain $k = 4$ congruent pieces similar to each piece of the previous step, and the magnification factor is $M = 3$, so the fractal dimension is

$$D = \frac{\log k}{\log M} = \frac{\log 4}{\log 3} = 1.261859507\ldots.$$

(c) [5 marks] Describe in your own words how the chaos game can be played to generate the fractal $\mathcal{A}$.

Solution: Start with a point x_0 in the plane. Pick one of A_1, A_2, A_3 or A_4 and apply it to x_0 to obtain x_1. Now pick one of A_1, A_2, A_3 or A_4 and apply it to x_1 to obtain x_2. Continue recursively in this way: to obtain x_{k+1} randomly pick either A_1, A_2, A_3 or A_4 and apply it to x_k. The orbit of x_0, namely $x_0, x_1, x_2, \ldots, x_k, \ldots$ as $k \to \infty$, is attracted to $\mathcal{A}$.

2. [20 marks] This question has four parts.

(a) [4 marks] Define the Mandelbrot Set, $\mathcal{M}$.

Solution: the actual definition on page 249 of Devaney is

$\mathcal{M}$ consists of all c-values for which the filled Julia set K_c is connected.

I would also accept this equivalent statement:

$$\mathcal{M} = \{c \in \mathbb{Z} \mid \text{the orbit of } 0 \text{ under } Q_c \text{ is bounded}\}$$

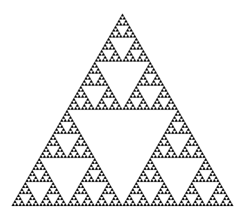
(b) [6 marks] Define the Sierpinski triangle. What is its fractal dimension?

Solution: here's a recursive definition:

1. Start with an equilateral triangle.
2. Remove the equilateral triangle that joins the midpoints of the three sides of the triangle.
3. Remove the equilateral triangle that joins the midpoints of the three sides in the remaining three triangles.
4. Repeat this process: remove the equilateral triangle that joins the midpoints of the three sides in each of the remaining triangles from the previous step.
5. The Sierpinski triangle is the set of points remaining in the limit as this process is repeated over and over without end.

The fractal dimension of the Sierpinski triangle is

$$\frac{\log 3}{\log 2} = 1.584962501\ldots$$



(c) [5 marks] Define what it means, according to Devaney, for $F : X \longrightarrow X$ to be chaotic.

Solution: Devaney's definition has three properties.

I. The periodic points of F are dense in X: for any point x in X and any $\epsilon > 0$ there is a periodic point of F within ϵ of x.

II. F is transitive: for any pair of points x and y in X and any $\epsilon > 0$ there is a third point z in X such that z is within ϵ of x and the orbit of z under F comes within ϵ of y.

III. F depends sensitively on initial conditions: there is a $\beta > 0$ such that for any x in X and any $\epsilon > 0$ there is a y in X such that y is within ϵ of x and for some k the distance between $F^k(x)$ and $F^k(y)$ is at least β.

(d) [5 marks] Prove that if $\mathbf{s} \in \Sigma$ then there is a sequence $\mathbf{t} \in \Sigma$ arbitrarily close to $\mathbf{s}$ for which $d[\sigma^n(\mathbf{s}), \sigma^n(\mathbf{t})] = 2$, for all sufficiently large n.

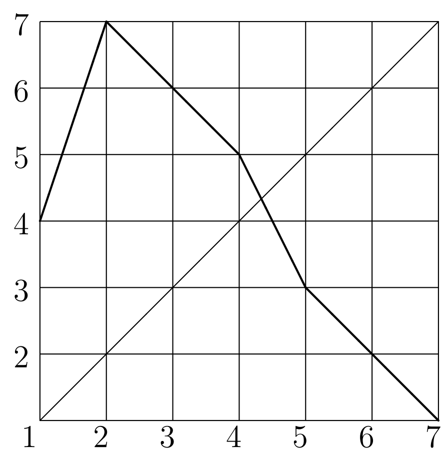
Solution: pick k large enough so that $1/2^k < \epsilon$. Let

$$t_m = \begin{cases} s_m, & \text{if } m \leq k, \\ 1, & \text{if } m > k \text{ and } s_m = 0, \\ 0, & \text{if } m > k \text{ and } s_m = 1 \end{cases}$$

By the Proximity Theorem $d[\mathbf{s}, \mathbf{t}] < 1/2^k < \epsilon$; but if $n > k$ then

$$d[\sigma^n(\mathbf{s}), \sigma^n(\mathbf{t})] = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2.$$

3. [20 marks] The graph of $F : [1, 7] \longrightarrow [1, 7]$ is shown below, along with the line $y = x$.



(a) [5 marks] Show that 1 is on a 7-cycle for F.

Solution: the 7-cycle for F is

$$1 \to 4 \to 5 \to 3 \to 6 \to 2 \to 7 \to 1$$

(b) [5 marks] Explain why F has cycles with prime period p for any even number p and for any odd number $p \geq 7$.

Solution: by Sarkovskii's Theorem F will have additional cycles of prime period p for any number p after 7 in the Sarkovskii ordering. But these numbers are all the odd numbers bigger than 7, all the even numbers, and 1.

(c) [10 marks] Prove that F has no cycle of prime period $p = 3$ or 5.

Solution: By Sarkovskii's Theorem we need only show that F has no 5-cycle. To this end, consider F on each of the six intervals $[1,2], [2,3], \ldots, [6,7]$ and show that F cannot have a point of period 5 in any of them.

[1,2]: under F,

$$[1,2] \longrightarrow [4,7] \longrightarrow [1,5] \longrightarrow [3,7] \longrightarrow [1,6] \longrightarrow [2,7],$$

so if $x \in [1,2]$ and $F^5(x) = x$ we must have $x = 2$. But this contradicts the fact that 2 is on a 7-cycle.

[2,3]: under F,

$$[2,3] \longrightarrow [6,7] \longrightarrow [1,2] \longrightarrow [4,7] \longrightarrow [1,5] \longrightarrow [3,7],$$

so if $x \in [2,3]$ and $F^5(x) = x$ we must have $x = 3$. But this contradicts the fact that 3 is on a 7-cycle.

[3,4]: under F,

$$[3,4] \longrightarrow [5,6] \longrightarrow [2,3] \longrightarrow [6,7] \longrightarrow [1,2] \longrightarrow [4,7],$$

so if $x \in [3,4]$ and $F^5(x) = x$ we must have $x = 4$. But this contradicts the fact that 4 is on a 7-cycle.

[4,5]: under F,

$$[4,5] \longrightarrow [3,5] \longrightarrow [3,6] \longrightarrow [2,6] \longrightarrow [2,7] \longrightarrow [1,7] \supset [4,5].$$

So we need a different argument. Let H be F restricted to $[4,5]$; calculate $H(x) = -2x + 13$, with fixed point $x = 13/3$. As H^5 is one-to-one, the fixed point of H will be the only solution to $H^5(x) = x$. (Could compute: $H^5(x) = -32x + 143$ and $H^5(x) = x \Leftrightarrow x = 13/3 \Leftrightarrow H(x) = x$.)

[5,6]: under F,

$$[5,6] \longrightarrow [2,3] \longrightarrow [6,7] \longrightarrow [1,2] \longrightarrow [4,7] \longrightarrow [1,5],$$

so if $x \in [5,6]$ and $F^5(x) = x$ we must have $x = 5$. But this contradicts the fact that 5 is on a 7-cycle.

[6,7]: under F,

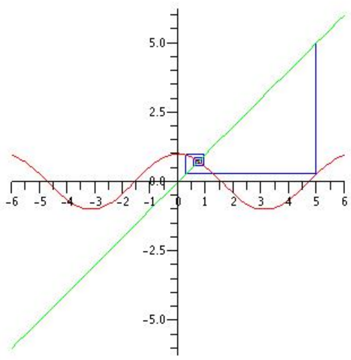
$$[6,7] \longrightarrow [1,2] \longrightarrow [4,7] \longrightarrow [1,5] \longrightarrow [3,7] \longrightarrow [1,6],$$

so if $x \in [6,7]$ and $F^5(x) = x$ we must have $x = 6$. But this contradicts the fact that 6 is on a 7-cycle.

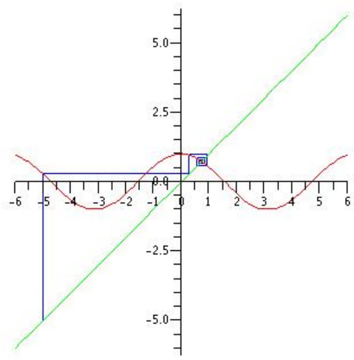
4. [20 marks] For $c \neq 0$, let $F_c : \mathbb{R} \longrightarrow \mathbb{R}$ by $F_c(x) = c \cos x$.

(a) [5 marks] Show that for $|c| < 1$, F_c has one fixed point and its basin of attraction is $\mathbb{R} = (-\infty, \infty)$.

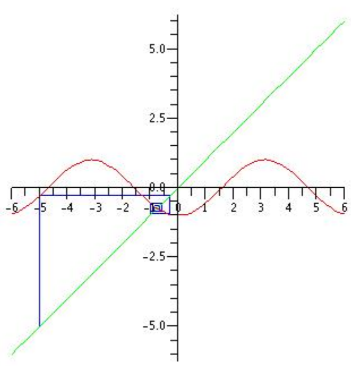
Solution: for $|c| < 1$ the graph of $F_c(x)$ intersects the line $y = x$ at one single fixed point p in the interval $(-\pi/2, \pi/2)$ and $|F_c'(p)| = |-c \sin p| \leq |c| < 1$, so this fixed point is attracting. Use graphical analysis to show that for $x_0 \in \mathbb{R}$, $x_n \longrightarrow p$.



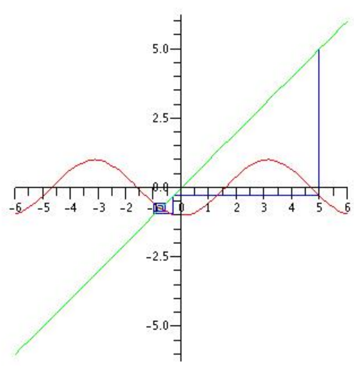
orbit of 5 under $c \cos x, 0 < c < 1$



orbit of -5 under $c \cos x, 0 < c < 1$



orbit of -5 under $c \cos x, -1 < c < 0$



orbit of 5 under $c \cos x, -1 < c < 0$

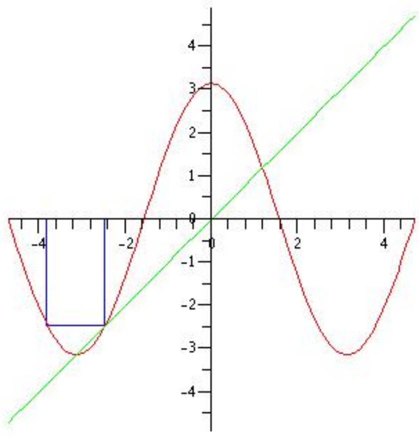
(b) [4 marks] Calculate the Schwarzian derivative of F_c and show it is negative.

Solution: for $\sin x \neq 0$, $S(F_c)(x) =$

$$\frac{F_c'''(x)}{F_c'(x)} - \frac{3}{2}\left(\frac{F_c''(x)}{F_c'(x)}\right)^2 = \frac{c \sin x}{-c \sin x} - \frac{3}{2}\left(\frac{-c \cos x}{-c \sin x}\right)^2 = -1 - \frac{3}{2}\cot^2 x < 0.$$

(c) [5 marks] Give a graphical example of a fixed point of F_c for which the immediate basin of attraction does not extend to infinity.

Solution: Let $c = \pi$, then $p = -\pi$ is a strongly attracting fixed point of F_π since



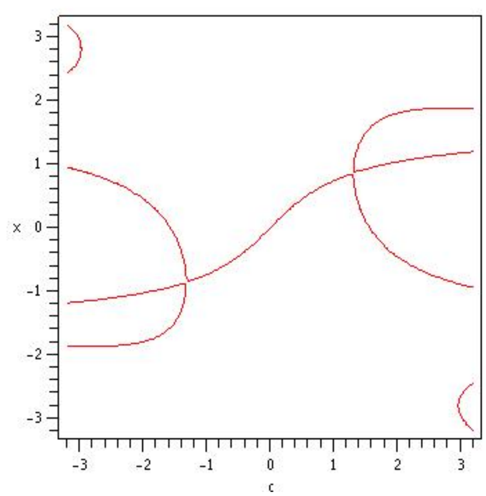
$$F_\pi(-\pi) = \pi \cos \pi = -\pi$$

and

$$F'_\pi(\pi) = -\pi \sin \pi = 0.$$

(This is Devaney's example in Sec 12.2.) The immediate basin of attraction of the fixed point $-\pi$ is indicated in blue on the graph to the left. It is the finite interval (a, b) where b is the repelling fixed point of F_π, close to -2.5, and a is the pre-image of b, close to -3.8, that satisfies $F_\pi(a) = F_\pi(b)$; namely $a = -2\pi - b$.

(d) [6 marks] Below is part of the bifurcation diagram for F_c, for $|c| < 3.2$, $|x| < 3.2$.



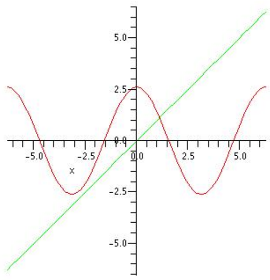
Classify each node in this diagram as a tangent (or saddle-node) bifurcation, a period-doubling bifurcation, or neither.

Solution: If p is a fixed point of $F_c(x)$ then

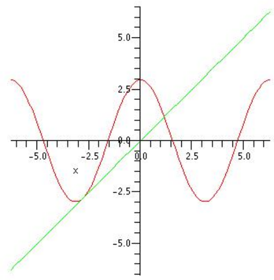
$$F_{-c}(-p) = -c\cos(-p) = -c\cos p = -F_c(p) = -p.$$

Thus (c, p) is a node on the bifurcation diagram if and only if $(-c, -p)$ is as well, and they both will be the same type of node. So we only need to classify the two nodes for which $c > 0$.

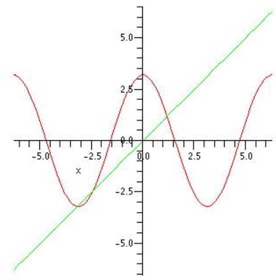
nodes $\pm(c,p), c$ **close to 3:** these are both saddle nodes, or tangent bifurcations. Graphically, compare graphs of $F_c(x)$ and $y = x$ for c close to 3:



$c = 2.6$



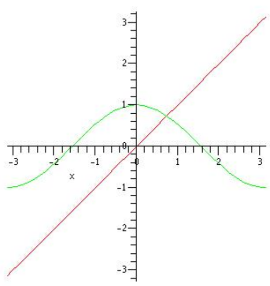
$c = 2.9\ldots$



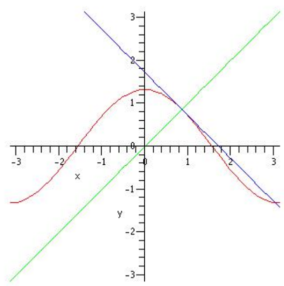
$c = 3.2$

You can see that the left half of the cosine graph approaches the line $y = x$, is tangent to it, and then dips below it, producing two fixed points, one attracting and one repelling. (Aside: this was a question in Problem Set 3.)

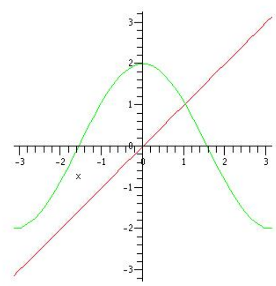
nodes $\pm(c,p), c$ **close to 1.3:** these are both period doubling bifurcations.



$c = 1$



$c = 1.32\ldots$

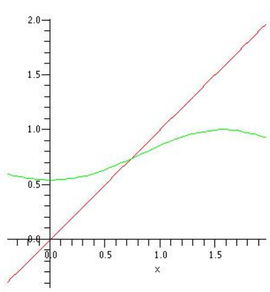


$c = 2$

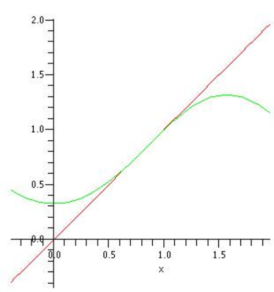
The above graphs illustrate how the attracting fixed point of F_c for $0 < c < 1$ becomes neutral, around $c = 1.32$, and then becomes repelling for $c > 1.32$. We claim an attracting 2-cycle appears for $c > 1.32$. For example consider $F_{\pi/2}$:

$$F_{\pi/2}(0) = \frac{\pi}{2}\cos 0 = \frac{\pi}{2}; F_{\pi/2}(\pi/2) = \frac{\pi}{2}\cos(\pi/2) = 0.$$

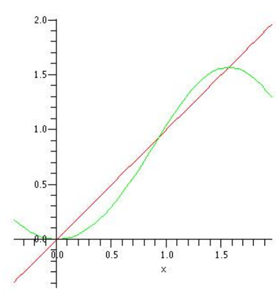
This 2-cycle is actually strongly attracting since $(F^2_{\pi/2})'(0) = 0$. For interest here are the graphs of $F_c^2(x)$ for $c = 1, c = 1.32, c = \pi/2$ on the interval $[0, \pi/2]$.



$c = 1$



$c = 1.32\ldots$



$c = \pi/2$

5. [20 marks] Determine the fate of the orbits of the following seeds z_0 under the following functions F. If the orbit is periodic, or eventually periodic, determine if the periodic cycle is attracting, repelling or neutral.

(a) [4 marks] $z_0 = \dfrac{3}{10}$ and $F(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1. \end{cases}$

Solution: this is the doubling function.

$$\frac{3}{10} \longrightarrow \frac{3}{5} \longrightarrow \frac{1}{5} \longrightarrow \frac{2}{5} \longrightarrow \frac{4}{5} \longrightarrow \frac{3}{5},$$

so the orbit of $3/10$ is attracted to a 4-cycle. This 4-cycle is repelling, since

$$|F'(3/15)F'(1/5)F'(2/5)F'(4/5)| = 2^4 = 16 > 1.$$

(b) [4 marks] $z_0 = 1$ and $F(z) = \dfrac{iz}{2}$.

Solution: observe that $F(0) = 0$ and $F'(z) = i/2$.

$$1 \longrightarrow \frac{i}{2} \longrightarrow -\frac{1}{4} \longrightarrow -\frac{i}{8} \longrightarrow \frac{1}{16} \longrightarrow \cdots \longrightarrow 0,$$

so the orbit of 1 is attracted to the fixed point 0, which is attracting since

$$|F'(0)| = |i/2| = 1/2 < 1.$$

(c) [4 marks] $z_0 = 0$ and $F(z) = z^2 + i$.

Solution: observe that $F'(z) = 2z$.

$$0 \longrightarrow i \longrightarrow -1+i \longrightarrow -i \longrightarrow -1+i \longrightarrow -i \longrightarrow \cdots,$$

so the orbit of 0 is attracted to the 2-cycle $-i, -1+i$ which is repelling since

$$|F'(-i)F'(-1+i)| = |(-2i)(-2+2i)| = |4-4i| = 4\sqrt{2} > 1.$$

(d) [4 marks] $z_0 = 0$ and $F(z) = z^2 + 2i - 1$.

Solution: observe that $F = Q_{2i-1}$ and that $|2i - 1| = \sqrt{5} > 2$. By the escape criterion, the orbit of 0 under Q_{2i-1} will be unbounded. That is, the orbit goes to infinity.

(e) [4 marks] $z_0 = 0$ and $F(z) = z^2 + \dfrac{i}{8} - 1$.

Solution: observe that $F = Q_{i/8-1}$ and that $|i/8 - 1 + 1| = |i/8| = 1/8 < 1/4$. So $c = i/8 - 1$ is in the period-2 bulb, $\{c \in \mathbb{C} \mid |c+1| < 1/4\}$, of the Mandelbrot set. Hence the orbit of 0 under $Q_{i/8-1}$ will be attracted to an attracting 2-cycle.

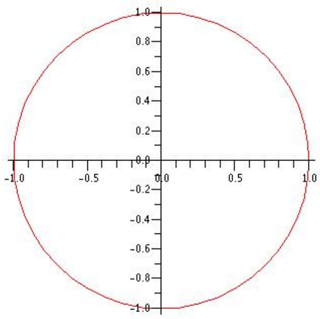
6. [20 marks] Let $Q_c : \mathbb{C} \longrightarrow \mathbb{C}$ by $Q_c(z) = z^2 + c$. Let K_c be the filled Julia set of Q_c; let J_c be the Julia set of Q_c.

(a) [5 marks] Plot K_0 and J_0 in the complex plane.

Solution: $Q_0(z) = z^2$ and $K_0 = \{z \in \mathbb{C} \mid \text{orbit of } z \text{ under } Q_0 \text{ is bounded}\}$. If $z = re^{i\theta}$, then

$$Q_0^n(z) = r^{2^n} e^{2^n i\theta} \text{ and } |Q_0^n(z)| = r^{2^n}.$$

So the orbit of z under Q_0 is unbounded if and only if $r > 1$. Hence



K_0 is the unit disc, and J_0 is its boundary, the unit circle.

$$K_0 = \{z \in \mathbb{C} \mid |z| \leq 1\};$$

$$J_0 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

(b) [10 marks] Let $R = \{z \in \mathbb{C} \mid |z| > 1\}$; let $H : R \longrightarrow \mathbb{C} - [-2, 2]$ by

$$H(z) = z + \frac{1}{z}.$$

Show that H is a conjugacy between Q_0 on R and Q_{-2} on $\mathbb{C} - [-2, 2]$.

Solution: we need to show H is a homeomorphism and that $Q_{-2} \circ H = H \circ Q_0$. Checking the equality first:

$$\begin{aligned}
Q_{-2}(H(z)) &= Q_{-2}(z + z^{-1}) \\
&= (z + z^{-1})^2 - 2 \\
&= z^2 + 2 + z^{-2} - 2 \\
&= z^2 + z^{-2} \\
&= H(z^2) \\
&= H(Q_0(z))
\end{aligned}$$

$$\begin{tikzcd}
R \arrow[r, "Q_0"] \arrow[d, "H"'] & R \arrow[d, "H"] \\
\mathbb{C} - [-2,2] \arrow[r, "Q_{-2}"'] & \mathbb{C} - [-2,2]
\end{tikzcd}$$

To show H is a homeomorphism, we need to show:

1. H is one-to-one,
2. H is onto,
3. H is continuous,
4. H^{-1} is also continuous.

H **is one-to-one:** suppose $z, w \in R$. Then

$$\begin{aligned} H(z) = H(w) \quad &\Rightarrow \quad z + \frac{1}{z} = w + \frac{1}{w} \\ &\Rightarrow \quad z^2 w + zw = zw^2 + zw \\ &\Rightarrow \quad zw(z - w) = 0 \\ &\Rightarrow \quad z = w \text{ or } z = w^{-1}. \end{aligned}$$

But if $|w| > 1$, then $|w^{-1}| < 1$; so $z = w$.

H **is onto:** let $w \in \mathbb{C} - [-2, 2]$ and let $H(z) = w$. Then

$$z + \frac{1}{z} = w \Rightarrow z^2 - wz + 1 = 0,$$

which is a quadratic equation with two solutions, z_1 and z_2 such that $z_1 z_2 = 1$. If $z_1 = e^{i\theta}$, then $z_2 = z_1^{-1} = e^{-i\theta}$ and more importantly,

$$H(z_1) = e^{i\theta} + e^{-i\theta} = 2\cos\theta \in [-2, 2].$$

So $|z_1| \neq 1$, which means one of z_1 or z_2 is in R.

H **is continuous:** since $z \neq 0$ for all $z \in R$ and

$$H'(z) = 1 - \frac{1}{z^2}.$$

It follows that H is differentiable for all $z \in R$, from which continuity follows.

H^{-1} **is continuous:** let $G = H^{-1}$, which exists by 1, 2 above. Let $w \in \mathbb{C} - [-2, 2]$. Then $G(w) \in R \Rightarrow |G(w)| > 1 \Rightarrow H'(G(w)) \neq 0$. Then

$$G'(w) = \frac{1}{H'(G(w))}$$

exists for all $w \in \mathbb{C} - [-2, 2]$ and G is differentiable, hence continuous.

(c) [5 marks] Plot K_{-2} and J_{-2} in the complex plane.

Solution: we did this in class. $K_{-2} = [-2, 2]$, which is its own boundary in $\mathbb{C}$ so $J_{-2} = K_{-2}$ as well. More formally, use the conjugacy of part (b) and the result of part (a). By part (a), the orbit of z under Q_0 tends to infinity for any $z \in R$; so by conjugacy, the orbit of $H(z)$ under Q_{-2} tends to infinity for all $H(z) \in \mathbb{C} - [-2, 2]$. Thus the only bounded orbits under Q_{-2} will be orbits of $w \in [-2, 2]$, whence

$$K_{-2} = [-2, 2].$$