University of Toronto
SOLUTIONS to **MAT335H1F TERM TEST**
**Monday, October 18, 2010**
Duration: 50 minutes

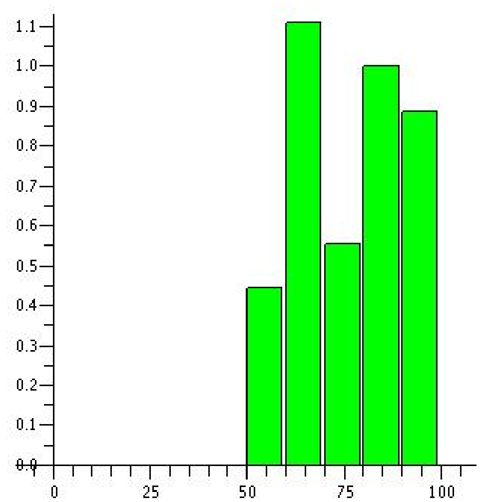
**Only aids permitted:** Scientific calculator, to be supplied by student.

**General Comments about the Test:**

- The average was good and nobody failed.
- For 1(d), many students missed the connection between $\Gamma$ and $K$ entirely.
- There were substantial difficulties with the algebra in Question 2.
- Remember: when solving
$$F^2(x) - x = 0$$
the expression
$$F(x) - x$$
must be a factor!

**Breakdown of Results:** 36 students wrote this test. The marks ranged from 54% to 98%, and the average was 76%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | % | Decade | % |
|---|---|---|---|
| | | 90-100% | 22.2 % |
| A | 47.2 % | 80-89% | 25.0 % |
| B | 13.9 % | 70-79% | 13.9 % |
| C | 27.8 % | 60-69% | 27.8 % |
| D | 11.1 % | 50-59% | 11.1 % |
| F | 0.0 % | 40-49% | 0.0 % |
| | | 30-39% | 0.0 % |
| | | 20-29% | 0.0 % |
| | | 10-19% | 0.0 % |
| | | 0-9% | 0.0 % |



1. [25 marks] Let $T(x) = \begin{cases} 3x, & \text{if } x \leq 1/2 \\ 3-3x, & \text{if } x > 1/2 \end{cases}$

(a) [5 marks] Find the fixed points of $T$ and determine if they are attracting or repelling.

**Solution:** If $x < 1/2$, then

$$T(x) = x \Leftrightarrow 3x = x \Leftrightarrow x = 0.$$

If $x > 1/2$, then

$$T(x) = x \Leftrightarrow 3 - 3x = x \Leftrightarrow 3 = 4x \Leftrightarrow x = \frac{3}{4}..$$

Both fixed points are repelling since

$$T'(0) = 3 > 1 \text{ and } T'(3/4) = -3 < -1.$$

(b) [10 marks] 3/13 is on a 3-cycle for $T$. What is the 3-cycle? Is it repelling or attracting?

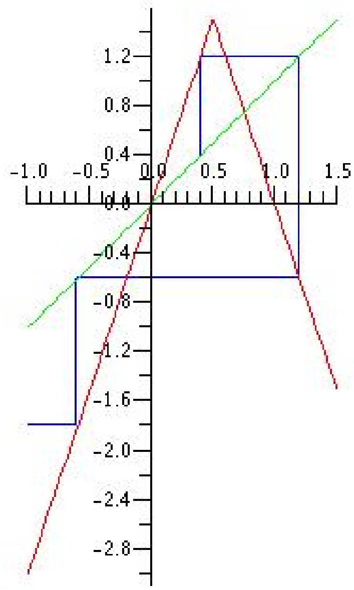
**Solution:**

$$T(3/13) = \frac{9}{13};\ T(9/13) = 3 - \frac{27}{13} = \frac{12}{13};\ T(12/13) = 3 - \frac{36}{13} = \frac{3}{13}.$$

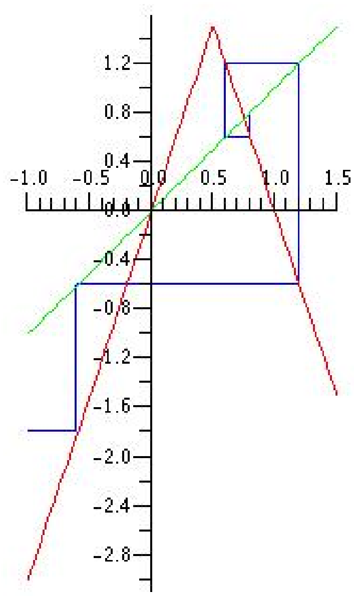
The 3-cycle is repelling since

$$T'(3/13)T'(9/13)T'(12/13) = (3)(-3)(-3) = 27 > 1.$$

(c) [5 marks] On the two graphs below, use graphical analysis to show that the orbits of both $x = 0.4$ and $y = 0.8$ under $T$ go to $-\infty$.




The orbit of $x = 0.4$ under $T$ goes to $-\infty$




The orbit of $y = 0.8$ under $T$ goes to $-\infty$

(d) [5 marks] Explain briefly and clearly why

$$\Gamma = \{x \in [0,1] \mid T^n(x) \in [0,1] \text{ for all } n\}$$

is equal to the Cantor middle-thirds set. (You do not have to prove your statements, and you do not need to do anything with ternary expansions!)

**Solution:** List the key connections:

1. The orbit of $x$ under $T$ goes to $-\infty$ if and only if $x$ is in one of the middle-thirds,

$$(1/2, 2/3), (1/9, 2/9), (7/9, 8/9), \ldots$$

2. 

$$\begin{aligned} \Gamma &= \{x \in [0,1] \mid T^n(x) \in [0,1] \text{ for all } n\} \\ &= [0,1] - \{x \in [0,1] \mid T^n(x) \to -\infty\} \end{aligned}$$

3. So $\Gamma = K$, the Cantor middle-thirds set, by definition.

2. [25 marks] Consider the family of functions $F_c(x) = x^2 + cx$, with parameter $c$.

(a) [8 marks] Find the fixed points of $F_c$ and determine for which values of $c$ they are attracting.

**Solution:**

$$F_c(x) = x \Leftrightarrow x^2 + cx = x \Leftrightarrow x(x + c - 1) = 0 \Leftrightarrow x = 0 \text{ or } x = 1 - c.$$

$$F_c'(x) = 2x + c.$$

So

$$F'(0) = c$$

and $x = 0$ is attracting if and only if $-1 < c < 1$. Also:

$$F'(1-c) = 2 - 2c + c = 2 - c;$$

so $x = 1 - c$ is attracting if and only if

$$|2 - c| < 1 \Leftrightarrow -1 < 2 - c < 1 \Leftrightarrow -3 < -c < -1 \Leftrightarrow 3 > c > 1.$$

(b) [6 marks] Find the points of prime period 2 for $F_c$.

**Solution:**

$$\begin{aligned}
F_c^2(x) = x \quad &\Leftrightarrow \quad (x^2 + cx)^2 + c(x^2 + cx) = x \\
&\Leftrightarrow \quad x^4 + 2cx^2 + c^2x^2 + cx^2 + c^2x - x = 0 \\
&\Leftrightarrow \quad x^4 + 2cx^2 + (c^2 + c)x^2 + (c^2 - 1)x = 0 \\
&\Leftrightarrow \quad (x^2 + (c-1)x)(x^2 + (1+c)x + 1 + c) = 0 \\
&\Leftrightarrow \quad x = 0, x = 1 - c, \text{ or } x = \frac{-(1+c) \pm \sqrt{(1+c)^2 - 4(1+c)}}{2} \\
&\Leftrightarrow \quad x = 0, x = 1 - c, \text{ or } x = \frac{-1 - c \pm \sqrt{c^2 - 2c - 3}}{2}
\end{aligned}$$

So the points of prime period 2 for $F_c$ are

$$p = \frac{-1 - c - \sqrt{c^2 - 2c - 3}}{2} \text{ and } q = \frac{-1 - c + \sqrt{c^2 - 2c - 3}}{2},$$

which only exist if

$$c^2 - 2c - 3 \geq 0 \Leftrightarrow (c-3)(c+1) \geq 0 \Leftrightarrow c \leq -1 \text{ or } c \geq 3.$$

(c) [4 marks] Describe the bifurcation at $c = 1$.

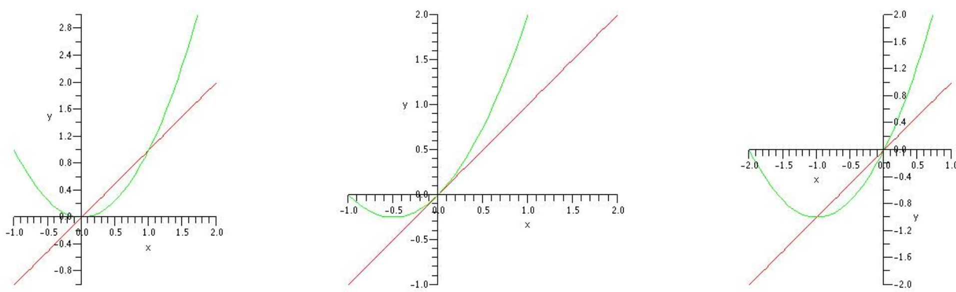
Three possible ways to present your solution:

**Analytically:** For $-1 < c < 3$ and $-2 < x < 3$ we have:

1. for $-1 < c < 1$ the fixed point $x = 0$ is attracting but the fixed point $x = 1 - c$ is repelling;
2. for $c = 1$ there is only one fixed point, $x = 0$, which is neutral;
3. for $1 < c < 3$ the fixed point $x = 0$ is repelling but the fixed point $x = 1 - c$ is attracting.

So there is no tangent bifurcation or period doubling bifurcation at $c = 1$.

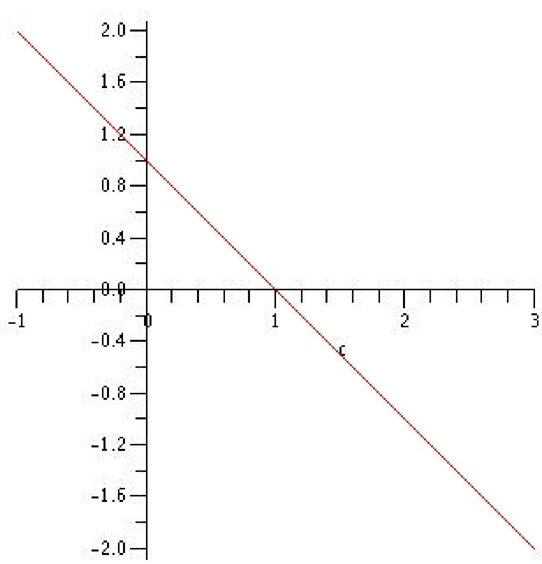
**Graphically:** $F_c'(0) = 1 \Leftrightarrow c = 1$; so there is tangency for $c = 1$.



$c < 1$ $\qquad$ $c = 1$ $\qquad$ $c > 1$

But for both $c < 1$ and $c > 1$ there are two fixed points, one attracting and one repelling. So there is no tangent bifurcation at $c = 1$.

**Bifurcation Diagram:** For $-1 < c < 3$ and $-2 < x < 3$ the two fixed points $x = 0$ and $x = 1 - c$ swap attracting and repelling roles at $c = 1$:



(d) [7 marks] Describe the bifurcation at $c = 3$.

Again, at least three approaches are possible:

**Analytically:**

$$\begin{aligned} |F_c'(p)F_c'(q)| &= |(-1-\sqrt{c^2-2c-3})(-1+\sqrt{c^2-2c-3})| \\ &= |1-(c-3)(c+1)| \\ &= |c^2-2c-4| \\ &= |(c-1)^2-5| \end{aligned}$$

Hence the 2-cycle $p, q$ is attracting, for $c > 3$, if

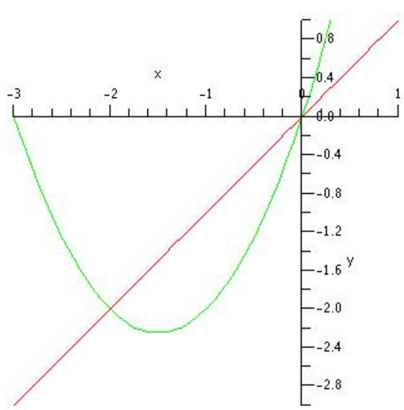
$$-1 < (c-1)^2-5 < 1 \Leftrightarrow 4 < (c-1)^2 < 6 \Leftrightarrow 2 < c-1 < \sqrt{6} \Leftrightarrow 3 < c < 1+\sqrt{6},$$

where we have used the fact that $c > 3$ to set $|c-1| = c-1$. So

1. for $2 < c < 3$ the fixed point $x = 1-c$ is attracting, and there is no 2-cycle;
2. for $c = 3$ the fixed point, $x = -2$ is neutral; and $p = q = -2$;
3. for $3 < c < 1+\sqrt{6}$ the fixed point $x = 1-c$ is repelling and the 2-cycle $p, q$ is attracting.

There is a period doubling bifurcation at $c = 3$.

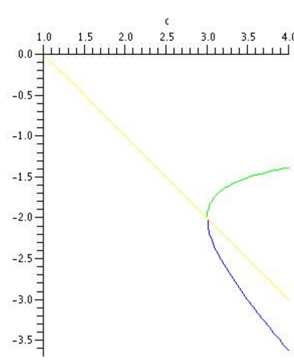
**Graphically:**
$F_c'(1-c) = -1 \Leftrightarrow 2-c = -1 \Leftrightarrow c = 3$.




So $F_3(x)$ is normal to the line $y = x$ at $x = -2$, and $F_c(x)$ has a period doubling bifurcation at $c = 3$. (See Note 3 on page 62 of Devaney.)

**Bifurcation Diagram:**
$x = 1-c$ is the yellow line; the blue ($p$) and green ($q$) curves are the 2-cycle

$$\frac{-1-c \pm \sqrt{c^2-2c-3}}{2}.$$




An attracting 2-cycle appears as the attracting fixed point becomes repelling.